Mathematics 1174

SOLUTIONS TO SELECTED EXERCISES FROM MATH 1174 TEXTBOOK

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Chapter 1

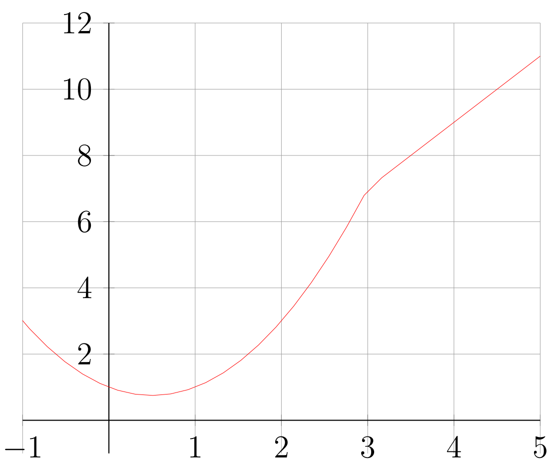
Limits

1.1 An Introduction to Limits

13. Let $f(x) = \begin{cases} x^2 - x + 1, & x \leq 3, \\ 2x + 1, & x > 3. \end{cases}$

x	$f(x)$
2.99	6.95010
2.999	6.99501
2.9999	6.99950
$\vdots$	$\vdots$
3.01	7.05010
3.001	7.00501
3.0001	7.00050
$\vdots$	$\vdots$

So, from the table above and the graph below, we can see that the $\lim_{x \to 3} f(x) = 7$.



17. Here, we have $f(x) = 9x + 0.06$, $a = -1$. From the table below, we see that the limit seems to be exactly 9.

h	$\frac{f(-1+h)-f(-1)}{h}$
-0.1	9
-0.01	9
$\vdots$	$\vdots$
0.01	9
0.1	9
$\vdots$	$\vdots$

19. Here, we have $f(x) = \frac{1}{x+1}$, $a = 2$. From the table below, we see that the limit is approximately -0.11.

h	$\frac{f(2+h)-f(2)}{h}$
-0.1	-0.114943
-0.01	-0.111483
$\vdots$	$\vdots$
0.01	-0.110742
0.1	-0.107527
$\vdots$	$\vdots$

21. Here, we have $f(x) = \ln x$, $a = 5$. From the table below, we see that the limit is approximately 0.2.

h	$\frac{f(5+h)-f(5)}{h}$
-0.1	0.202027
-0.01	0.2002
$\vdots$	$\vdots$
0.01	0.1998
0.1	0.198026
$\vdots$	$\vdots$

23. Here, we have $f(x) = \cos x$, $a = \pi$. From the table below, we see that the limit is approximately ± 0.005. However, if we continue on by increasing the digit 0, eventually we will get the limit approaching 0.

h	$\frac{f(\pi+h)-f(\pi)}{h}$
-0.1	-0.0499583
-0.01	-0.0499583
$\vdots$	$\vdots$
0.01	0.00499996
0.1	0.0499583
$\vdots$	$\vdots$

1.3 Finding Limits Analytically

7. $\lim_{x\to 9}\left[\frac{3f(x)}{g(x)}\right] = \frac{3\lim_{x\to 9} f(x)}{\lim_{x\to 9} g(x)} = \frac{3\cdot 6}{3} = 6.$

9. $\lim_{x\to 6}\left[\frac{f(x)}{3-g(x)}\right]$. This is not possible because as $x \to 6$, the denominator goes to zero. So, we can say that the limit in this particular case does not exit.

11. $\lim_{x\to 6} f(g(x))$. Not possible to know; as x approaches 6, $g(x)$ approaches 3, but we know nothing of the behaviour of $f(x)$ when x is near 3.

13. $\lim_{x\to 6}\left[f(x)g(x) - f^2(x) + g^2(x)\right] = \lim_{x\to 6} f(x)\cdot\lim_{x\to 6} g(x) - \lim_{x\to 6} f^2(x) + \lim_{x\to 6} g^2(x).$

    $= 9\cdot 3 - 81 + 9 = -45.$

15. $\lim_{x\to 10}\cos((g(x)) = \lim_{x\to 10}\cos(\pi) = -1.$

17. $\lim_{x\to 1} g(5f(x)).$

    $\lim_{x\to 1} 5f(x) = 5\lim_{x\to 1} f(x) = 10$ and $\lim_{x\to 10} g(x) = \pi$ and $g(10) = \pi.$

    $\Rightarrow \lim_{x\to 1} g(5f(x)) = \pi.$

19. $\lim_{x\to\pi}\left(\frac{x-3}{x-5}\right)^7 = \left(\frac{\pi-3}{\pi-5}\right)^7 \approx 0.$

25. $\lim_{x\to\pi}\frac{x^2+3x+5}{5x^2-2x-3} = \frac{\pi^2+3\pi+5}{5\pi^2-2\pi-3} \approx 0.6064.$

27. $\lim_{x\to 6}\frac{x^2-4x-12}{x^2-13x+42} = \lim_{x\to 6}\frac{(x-6)(x+2)}{(x-6)(x-7)} = \lim_{x\to 6}\frac{x+2}{x-7} = -8.$

29. $\lim_{x\to 2}\frac{x^2+6x-16}{x^2-3x+2} = \lim_{x\to 2}\frac{(x-2)(x+8)}{(x-2)(x-1)} = \lim_{x\to 2}\frac{x+8}{x-1} = 10.$

31. $\lim_{x\to -2}\frac{x^2-5x-14}{x^2+10x+16} = \lim_{x\to -2}\frac{(x-7)(x+2)}{(x+2)(x+8)} = \lim_{x\to -2}\frac{x-7}{x+8} = -\frac{3}{2}.$

1.4 One Sided Limits

5. a) $\lim_{x \to 1^-} f(x) = 2.$

   b) $\lim_{x \to 1^+} f(x) = 2.$

   c) $\lim_{x \to 1} f(x) = 2.$

   d) $f(1) = 1.$

   e) $\lim_{x \to 0^-} f(x)$. As f is not defined for $x < 0$, this limit is not defined.

   f) $\lim_{x \to 0^+} f(x) = 1.$

7. a) $\lim_{x \to 1^-} f(x) = +\infty$. Note: For now, we can also say the limit does not exist.

   b) $\lim_{x \to 1^+} f(x) = +\infty$. Note: For now, we can also say the limit does not exist.

   c) $\lim_{x \to 1} f(x) = +\infty$. Note: For now, we can also say the limit does not exist.

   d) $f(1)$ is not defined.

   e) $\lim_{x \to 2^-} f(x) = 0.$

   f) $\lim_{x \to 0^+} f(x) = 0.$

9. a) $\lim_{x \to 1^-} f(x) = 2.$

   b) $\lim_{x \to 1^+} f(x) = 2.$

   c) $\lim_{x \to 1} f(x) = 2.$

   d) $f(1) = 2.$

11. a) $\lim_{x \to -2^-} f(x) = 2.$

    b) $\lim_{x \to -2^+} f(x) = 2.$

c) $\lim_{x\to -2} f(x) = 2$.

d) $f(-2) = 0$.

e) $\lim_{x\to 2^-} f(x) = 2$.

f) $\lim_{x\to 2^+} f(x) = 2$.

g) $\lim_{x\to 2} f(x) = 2$.

h) $f(2)$ is not defined.

13. a) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x+1) = 2$.

b) $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2-5) = -4$.

c) $\lim_{x\to 1} f(x) = DNE$.

d) $f(1) = 2$.

15. a) $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (x^2-1) = 0$.

b) $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^-} (x^3+1) = 0$.

c) $\lim_{x\to -1} f(x) = 0$.

d) $f(-1) = 0$.

e) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^3+1) = 2$.

f) $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2+1) = 2$.

g) $\lim_{x\to 1} f(x) = 2$.

h) $f(1) = 2$.

17. a) $\lim_{x \to a^-} f(x) = \lim_{x \to a^-} (1 - \cos^2 x) = 1 - \cos^2 a$.

b) $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} \sin^2 x = \sin^2 a = 1 - \cos^2 a$.

c) $\lim_{x \to a} f(x) = 1 - \cos^2 a$.

d) $f(a) = \sin^2 a = 1 - \cos^2 a$.

19. a) $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 = 4$.

b) $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (-x^2 + 2x + 4) = 4$.

c) $\lim_{x \to 2} f(x) = 4$.

d) $f(2) = 2 + 1 = 3$.

21. a) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{-x}{x} = -1$.

b) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x}{x} = 1$.

c) $\lim_{x \to 0} f(x) = DNE$.

d) $f(0) = 0$.

1.5 Limits Involving Infinity

9. $f(x) = \dfrac{1}{(x+1)^2}$.

   a) $\lim_{x\to -1^-} f(x) = +\infty$.

   b) $\lim_{x\to -1^+} f(x) = +\infty$.

11. $f(x) = \dfrac{1}{e^x+1}$.

    a) $\lim_{x\to -\infty} f(x) = 1$.

    b) $\lim_{x\to +\infty} f(x) = 0$.

    c) $\lim_{x\to 0^-} f(x) = \dfrac{1}{2}$.

    d) $\lim_{x\to 0^+} f(x) = \dfrac{1}{2}$.

13. $f(x) = \cos x$.

    a) $\lim_{x\to -\infty} f(x)$ does not exist because $\cos x$ is bouncing between -1 and 1 for all x.

    b) $\lim_{x\to +\infty} f(x)$ does not exist because $\cos x$ is bouncing between -1 and 1 for all x.

15. $f(x) = \dfrac{x^2-1}{x^2-x-6} = \dfrac{(x-1)(x+1)}{(x-3)(x+2)}$.

    a) $\lim_{x\to 3^-} f(x) = \dfrac{(2)(4)}{(0^-)(5)} = \dfrac{8}{0^-} = -\infty$. Or we can try table of numbers like below

x	$f(x)$
2.9	-15.1224
2.99	-159.12
2.999	-1599.12
$\vdots$	$\vdots$

    b) $\lim_{x\to 3^+} f(x) = \dfrac{(2)(4)}{(0^+)(5)} = \dfrac{8}{0^+} = \infty$. Or we can try table of numbers like below

x	$f(x)$
3.1	16.8824
3.01	160.88
3.001	1600.88
$\vdots$	$\vdots$

    c) It seems that the limit $\lim_{x\to 3} f(x)$ does not exist.

17. $f(x) = \dfrac{x^2 - 11x + 30}{x^3 - 4x^2 - 3x + 18} = \dfrac{(x-6)(x-5)}{(x-2)(x-3)^2}$.

a) $\lim\limits_{x \to 3^-} f(x) = \dfrac{(-3)(-2)}{(1)(0^-)^2} = \dfrac{6}{0^+} = \infty$. Or we can try table of numbers like below

x	$f(x)$
2.9	132.857
2.99	12124.4
$\vdots$	$\vdots$

b) $\lim\limits_{x \to 3^+} f(x) = \dfrac{(-3)(-2)}{(1)(0^+)^2} = \dfrac{6}{0^+} = \infty$. Or we can try table of numbers like below

x	$f(x)$
3.1	108.039
3.01	11876.4
$\vdots$	$\vdots$

c) So, it seems that the limit $\lim\limits_{x \to 3} f(x) = \infty$.

19. $f(x) = \dfrac{2x^2 - 2x - 4}{x^2 + x - 20} = \dfrac{(x-2)(x+1)}{(x-4)(x+5)}$.

$\lim\limits_{x \to \infty} f(x) \approx \lim\limits_{x \to \infty} \dfrac{2x^2}{x^2} = 2$. So horizontal asymptote is $y = 2$.

$\lim\limits_{x \to 4^-} f(x) = \dfrac{(2)(5)}{(0^-)(9)} = \dfrac{10}{0^-} = -\infty$ and $\lim\limits_{x \to -5^-} f(x) = \dfrac{(-7)(-4)}{(-9)(0^+)} = \dfrac{10}{0^-} = +\infty$.

So vertical asymptotes are $x = 4$ and $x = -5$.

21. $f(x) = \dfrac{x^2 + x - 12}{7x^3 - 14x^2 - 21x} = \dfrac{(x-3)(x+4)}{x(x-3)(x+1)} = \dfrac{x+4}{x(x+1)}$.

$\lim\limits_{x \to \infty} f(x) \approx \lim\limits_{x \to \infty} \dfrac{x^2}{7x^3} = \lim\limits_{x \to \infty} \dfrac{1}{x} = 0$. So horizontal asymptote is $y = 0$.

$\lim\limits_{x \to 0^-} f(x) = \dfrac{4}{(0^-)(1)} = \dfrac{4}{0^-} = -\infty$ and $\lim\limits_{x \to -1^-} f(x) = \dfrac{3}{(-1)(0^-)} = \dfrac{4}{0^+} = +\infty$.

So vertical asymptotes are $x = 0$ and $x = -1$.

23. $f(x) = \dfrac{x^2 - 9}{9x + 27} = \dfrac{(x-3)(x+3)}{9(x+3)} = \dfrac{x-3}{9}, \quad x \neq -3$.

$\lim\limits_{x \to \infty} f(x) \approx \lim\limits_{x \to \infty} \dfrac{x^2}{9x} = \lim\limits_{x \to \infty} \dfrac{x}{9} = \infty$. So no horizontal asymptote.

The function does not have vertical asymptote as well because the denominator is 9.

25. $\lim_{x\to\infty} \dfrac{x^3+2x^2+1}{x-5} \approx \lim_{x\to\infty} \dfrac{x^3}{x} = \lim_{x\to\infty} x^2 = \infty.$

27. $\lim_{x\to-\infty} \dfrac{x^3+2x^2+1}{x^2-5} \approx \lim_{x\to\infty} \dfrac{x^3}{x^2} = \lim_{x\to-\infty} x = -\infty.$

1.6 Continuity

11. $f(x)$ is not continuous at $a = 1$ because

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 2 \text{ but } f(1) = 1.$$

13. $f(x)$ is not continuous at $a = 1$ because $f(1)$ is not defined or

$$\lim_{x\to 1^-} f(x) = \infty \text{ and } \lim_{x\to 1^+} f(x) = \infty.$$

15. $f(x)$ is continuous at $a = 1$ because

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1) = 2.$$

19. a) $f(x)$ is continuous at $a = 0$ because

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = 0^3 - 0 = 0.$$

b) $f(x)$ is not continuous at $a = 1$ because

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^3 - x = 0 \text{ and } \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x - 2 = -1 = f(1).$$

21. a) $f(x)$ is continuous at $a = 0$ because

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = \frac{0^2 - 64}{0^2 - 11\cdot 0 + 24} = -\frac{64}{24}.$$

b) $f(x)$ is not continuous at $a = 8$ because

$$\lim_{x\to 8^-} f(x) = \lim_{x\to 8^+} f(x) = \lim_{x\to 8} \frac{(x-8)(x+8)}{(x-8)(x-3)} = \lim_{x\to 8} \frac{x+8}{x-3} = \frac{16}{5}.$$

but $f(8) = 5$.

23. $g(x) = \sqrt{x^2 - 4}$.

So the domain of the function is $x^2 - 4 \geq 0 \Rightarrow x \in (-\infty, -2] \cup [2, +\infty)$.

The function f(x) is continuous at $x = -2$ and $x = 2$ because $\lim_{x\to -2^+} g(x) = 0$ and $\lim_{x\to 2^-} g(x) = 0$.

Therefore, the continuous interval is $(-\infty, -2] \cup [2, +\infty)$.

25. $g(x) = \sqrt{5t^2 - 30} = \sqrt{5(t^2 - 6)}$.

So the domain of the function is $x^2 - 4 \geq 0 \Rightarrow x \in (-\infty, -\sqrt{6}] \cup [\sqrt{6}, +\infty)$.

The function f(x) is continuous at $x = -\sqrt{6}$, $x = \sqrt{6}$, because $\lim_{x \to -\sqrt{6}^+} g(x) = 0$ and $\lim_{x \to \sqrt{6}^-} g(x) = 0$.

Therefore, the continuous interval is $(-\infty, -\sqrt{6}] \cup [\sqrt{6}, +\infty)$.

27. $g(x) = \dfrac{1}{1 + x^2}$.

The function g(x) is continuous everywhere, $(-\infty, +\infty)$.

31. $g(x) = 1 - e^k$.

We need $1 - e^k \geq 0 \Rightarrow 1 \geq e^k \Rightarrow k \leq 0$.

The function g(x) is continuous at x = 0 because $\lim_{x \to 0^+} g(x) = 0$.

Therefore, the continuous interval is $(-\infty, 0]$.

Chapter 2

Derivatives

2.1 Introduction to Derivative

7. $f(x) = 2x.$

   $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{2(x+h)-2x}{h} = \lim_{h\to 0} \frac{2h}{h} = 2.$$

9. $f(x) = x^2.$

   $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h}$$
   $$= \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0}(2x+h) = 2x.$$

11. $f(x) = \frac{1}{x}.$

   $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \lim_{h\to 0} \frac{x-(x+h)}{hx(x+h)}$$
   $$= \lim_{h\to 0} -\frac{1}{x(x+h)} = -\frac{1}{x^2}.$$

15. $f(x) = 4-3x$, at $x = 7$.

   $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{4-3(x+h)-(4-3x)}{h} = -3.$$

   At $x = 7 \Rightarrow y = f(7) = -17$. Then,

   a) the equation of the tangent line is $y+17 = -3(x-7)$.

   b) the equation of the normal line at the same point is $y+17 = \frac{1}{3}(x-7)$.

   Note: No need to simplify the results.

17. $f(x) = 3x^2 - x + 4$, at $x = -1$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{3(x+h)^2 - (x+h) + 4 - (3x^2 - x + 4)}{h}.$$

$$= \lim_{h\to 0} \frac{6xh + 3h^2 - h}{h} = 6x - 1.$$

At $x = -1 \Rightarrow y = f(-1) = 8, f'(-1) = -7$. Then,

a) the equation of the tangent line is $y - 8 = -7(x + 1)$.

b) the equation of the normal line at the same point is $y - 8 = \frac{1}{7}(x + 1)$.

Note: No need to simplify the results.

19. $f(x) = \frac{1}{x-2}$, at $x = 3$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h-2} - \frac{1}{x-2}}{h} = \lim_{h\to 0} \frac{x - 2 - x - h + 2}{h(x-2)(x+h-2)}.$$

$$= \lim_{h\to 0} \frac{-1}{(x-2)(x+h-2)} = \lim_{h\to 0} -\frac{1}{(x-2)^2}.$$

At $x = 3 \Rightarrow y = f(3) = 1, f'(3) = -1$. Then,

a) the equation of the tangent line is $y - 1 = -1 \cdot (x - 3)$.

b) the equation of the normal line at the same point is $y - 1 = 1 \cdot (x - 3)$.

Note: No need to simplify the results.

23. $f(x) = \frac{1}{x+1}$.

a) At $(0, 1), f'(0) \approx -1$. At $(1, 0.5), f'(1) \approx -0.25$.

b) $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h}$$
$$= \lim_{h\to 0} \frac{x + 1 - x - h - 1}{h(x+1)(x+h+1)} = \lim_{h\to 0} \frac{-1}{(x+1)(x+h+1)} = -\frac{1}{(x+1)^2}.$$

c) At $(0, 1), f'(0) = -1$. Then, the equation of the tangent line is $y - 1 = -1(x - 0) \Rightarrow y = -x + 1$.

At $(1, 0.5), f'(1) = -0.25$. Then, the equation of the tangent line is $y - 0.5 = -0.25(x - 1) \Rightarrow y = -0.25x + 0.75$.

2.3 Basic Differentiation Rules

9. $f(x) = 7x^2 - 5x + 7 \Rightarrow f'(x) = 14x - 5.$

11. $m(t) = 9t^5 - \frac{1}{8}t^3 + 3t - 8 \Rightarrow m'(t) = 45t^4 - \frac{3}{8}t^2 + 3.$

13. $p(s) = \frac{1}{4}s^4 + \frac{1}{3}s^3 + \frac{1}{2}s^2 + s + 1 \Rightarrow p'(s) = s^3 + s^2 + s + 1.$

15. Note that we will have a faster and better way of finding derivative of this type, but for now we expand the function to get the following:

    $g(x) = (2x-5)^3 = (2x-5)^2(2x-5) = (4x^2 - 20x + 25)(2x-5) = 8x^3 - 60x^2 + 150x + 125.$
    Thus, $g'(x) = 24x^2 - 120x + 150.$

17. Note: The same as the previous equation. $f(x) = (2-3x)^2 = 4-12x+9x^2 \Rightarrow f'(x) = -12+18x.$

19. First 4 derivatives of $h(t) = t^2 - e^t$.

    $\Rightarrow h'(t) = 2t - e^t \Rightarrow h''(t) = 2 - e^t \Rightarrow h'''(t) = -e^t \Rightarrow h^{(4)}(t) = -e^t.$

21. $f(x) = x^3 - x \Rightarrow f(1) = 0$. Also, $f'(x) = 3x^2 - 1 \Rightarrow f'(1) = 3 - 1 = 2.$

    Thus, the equation of the tangent line at $(1, 0)$ is $y - 0 = 2(x-1) \Rightarrow y = 2x - 2.$

    The slope of the normal line is $m = -\frac{1}{2}$. So, the equation of the normal line at $(1, 0)$ is $y - 0 = -\frac{1}{2}(x-1) \Rightarrow y = -\frac{1}{2}x + \frac{1}{2}.$

23. $f(x) = 2x + 3 \Rightarrow f(5) = 13$. Also, $f'(x) = 2 \Rightarrow f'(5) = 2.$

    Thus, the equation of the tangent line at $(5, 13)$ is $\Rightarrow y - 13 = 2(x-5) \Rightarrow y = 2x + 3$. Note that since the function is linear, the tangent line at any point is the line itself.

    The slope of the normal line is $m = -\frac{1}{2}$. So, the equation of the normal line at $(5, 13)$ is $\Rightarrow y - 13 = -\frac{1}{2}(x-5) \Rightarrow y = -\frac{1}{2}x + \frac{31}{2}.$

2.4 The Product and Quotient Rules

5. $f(x) = x(x^2 + 3x)$.

   a) $f(x) = x(x^2 + 3x) \Rightarrow f'(x) = 1(x^2 + 3x) + x(2x + 3) = 3x^2 + 6x$.

   b) $f(x) = x(x^2 + 3x) = x^3 + 3x^2 \Rightarrow f'(x) = 3x^2 + 6x$.

   c) We can see that they are equal.

7. $h(s) = (2s - 1)(s + 4)$.

   a) $h(s) = (2s - 1)(s + 4) \Rightarrow h'(s) = 2(s + 4) + (2s - 1) = 4s - 7$.

   b) $h(s) = (2s - 1)(s + 4) = 2s^2 - 7s - 4 \Rightarrow h'(s) = 4s - 7$.

   c) We can see that they are equal.

9. $f(x) = \dfrac{x^2 + 3}{x}$.

   a) $f(x) = \dfrac{x^2 + 3}{x} \Rightarrow f'(x) = \dfrac{2x(x) - (x^2 + 3)}{x^2} = \dfrac{x^2 - 3}{x^2} = 1 - 3x^{-2}$.

   b) $f(x) = \dfrac{x^2 + 3}{x} = x + 3x^{-1} \Rightarrow f'(x) = 1 - 3x^{-2}$.

   c) We can see that they are equal.

11. $h(s) = \dfrac{3}{4s^3}$

   a) $h(s) = \dfrac{3}{4s^3} \Rightarrow h'(s) = \dfrac{0 - 3(12s^2)}{16s^6} = -\dfrac{3}{4s^4} = -\dfrac{9}{4}s^{-4}$.

   b) $h(s) = \dfrac{3}{4s^3} = \dfrac{3}{4}s^{-3} \Rightarrow h'(s) = -\dfrac{9}{4}s^{-4}$.

   c) We can see that they are equal.

13. $g(x) = \dfrac{x + 7}{x - 5} \Rightarrow g'(x) = \dfrac{(x - 5) - (x + 7)}{(x - 5)^2} = -\dfrac{12}{(x - 5)^2}$.

15. $f(x) = \dfrac{x^4 + 2x^3}{x + 2} \Rightarrow f'(x) = \dfrac{(4x^3 + 6x^2)(x - 2) - (x^4 + 2x^3)}{(x + 2)^2}$. Note: No need to simplify the result.

17. $g(s) = e^s(s^2+2) \Rightarrow g'(s) = e^s(s^2+2) + e^s(2s) \Rightarrow g'(0) = 2.$

Thus, the equation of the tangent line at $(0, 2)$ is $\Rightarrow y - 2 = 2(s - 0) \Rightarrow y = 2s + 2.$

The slope of the normal line is $m = -\dfrac{1}{2}$. So, the equation of the normal line at $(0, 2)$ is $\Rightarrow y - 2 = -\dfrac{1}{2}(x - 0) \Rightarrow y = -\dfrac{1}{2}x + 2.$

19. The graph of $f(x)$has a horizontal tangent line at a point when $f'(x) = 0$. Here, we have $f(x) = 6x^2 - 18x - 24 \Rightarrow f'(x) = 12x - 18 = 0 \Rightarrow x = \dfrac{3}{2} \Rightarrow y = -\dfrac{75}{2}.$

21. Same as the previous question.
$f(x) = \dfrac{x^2}{x+1} \Rightarrow f'(x) = \dfrac{2x(x+1) - x^2}{(x+1)^2} = \dfrac{x^2+2x}{(x+1)^2} = \dfrac{x(x+2)}{(x+1)^2} = 0 \Rightarrow x = 0, -2.$

$x = 0 \Rightarrow y = 0.$

$x = -2 \Rightarrow y = -4.$

2.5 Derivative as Rates of Change

1. We have $s(t) = 2t^3 - 9t^2 + 12t$, where s is in cm and t is in s. Its velocity is $v(t) = 6t^2 - 18t + 12 = 6(t-1)(t-2)$. The particle is at rest when $v = 0 \Rightarrow t = 1, 2$. Here, we know that the particle changes its direction in its motion. When $v > 0$, it moves to the right, and $v < 0$, it moves to the left. Solving these inequalities, we get it moves to the right until $t = 1$; then moves to the left until $t = 2$; then it moves to the right again for $t > 2$. Note: When $a > 0$, it's accelerating to the right, and $a < 0$, it accelerates to the left. But, it speeds up when v and a have the same signs; it slows down when they have different signs.

3. $s(t) = 100t - 5t^2 \Rightarrow v(t) = 100 - 10t = 10(10 - t)$.

   a) $\bar{v} = \dfrac{s(4) - s(1)}{4 - 1} = 75$ m/s.

   b) $v = 0 \Rightarrow t = 10$ s. So, $s_{max} = s(10) = 500$ m.

   c) $v(2) = 80$ m/s and $v(15) = -50$ m/s

   d) $s_{total} = |s_{max} - s(5)| + |s(12) - s_{max}| = |125| + |20| = 145$ m.

5. $h(t) = 490 - 12gt^2$.

   a) $v(t) = \dfrac{h(2) - h(1)}{2 - 1} = \dfrac{19.6 - 372.4}{1} = -352.8$.

   b) $h(t+\epsilon) = 490 - 12g(t+\epsilon)^2 \Rightarrow v(t) = \dfrac{h(t+\epsilon) - h(t)}{t + \epsilon - t}$.

   $= \dfrac{490 - 12gt^2 - 24gt\epsilon + 12g\epsilon^2 - 490 + 12gt^2}{\epsilon} = \dfrac{-24gt\epsilon + 12g\epsilon^2}{\epsilon}$.

   $= -24gt + 12g\epsilon$

   c) $h(t) = 0 \Rightarrow 490 - 12gt^2 = 0 \Rightarrow t = \sqrt{\dfrac{490}{12 \cdot 9.8}} = 2.04$.

7. $R(x) = p \cdot x = (-3x^2 + 600x) \cdot x = -3x^3 + 600x^2 \Rightarrow R'(x) = -9x^2 + 1200x$.

   a) $MR(100) = 30,000$. This means that if the store increases sale from 100 to 101 cameras, revenue will increasing by $\$30,000$.

   b) $MR(300) = -450,000$. This means that if the store increases sale from 300 to 301 cameras, revenue will decreasing by $\$450,000$.

9. $C(x) = 4200 + 5.40x - 0.001x^2 + 0.000002x^3$.

   a) The average cost is given as $\bar{C}(x) = \dfrac{C(x)}{x} = \dfrac{4200}{x} + 5.40 - 0.001x + 0.000002x^2$. Then, $\bar{C}(1000) = \$10.60$ per unit.

   b) $MC(x) = 5.40 - 0.002x + 0.000006x^2 \Rightarrow MC(1000) = \$9.40$ per unit.

   c) $\bar{C}(1001) = \$10.5988$ per unit. The average cost will decrease by $\$0.0012$ per unit.

11. $p(x) = -3x^2 + 600x$ and $C(x) = 1800 + 357x^2$. We have
$R(x) = p{\cdot}x = (-3x^2+600x){\cdot}x = -3x^3+600x^2 \Rightarrow P(x) = R(x)-C(x) = -3x^3+243x^2-1800 \Rightarrow MP(x) = -9x^2 + 486x$.

   a) $MP(10) = 3960$. This means that if we increase the production from 10 to 11, we will increase the profit by \$3960.

   b) $MP(100) = -41,400$. This means that if we increase the production from 100 to 101, we will decrease the profit by \$41,400.

2.6 Derivative of Trigonometric Functions

3. $f(\theta) = 9\sin\theta + 10\cos\theta \Rightarrow f'(\theta) = 9\cos\theta - 10\sin\theta$.

5. $h(t) = e^t - \sin t - \cos t \Rightarrow h'(t) = e^t - \cos t + \sin t$.

7. $f(t) = \frac{1}{t^2}(\csc t - 4) \Rightarrow f'(t) = -\frac{2}{t^3}(\csc t - 4) - \frac{1}{t^2}\csc t \tan t$.

9. $h(x) = \cot x - e^x \Rightarrow h'(x) = -\csc^2 x - e^x$.

11. $f(x) = \dfrac{\sin x}{\cos x + 3} \Rightarrow f'(x) = \dfrac{\cos x(\cos x + 3) - \sin x(-\sin x)}{(\cos x + 3)^2} = \dfrac{1 + 3\cos x}{(\cos x + 3)^2}$.

13. $g(t) = 4t^3e^t - \sin t\cos t \Rightarrow g'(t) = (12t^2e^t + 4t^3e^t) - (\cos^2 t - \sin^2 t)$.

15. $f(x) = x^2e^x\tan x \Rightarrow f'(x) = (2xe^x + x^2e^x)\tan x + (x^2e^x \cdot \sec^2 x)$.

17. $f(x) = 4\sin x \Rightarrow f\left(\frac{\pi}{2}\right) = 4$, and also $f'(x) = 4\cos x \Rightarrow f'\left(\frac{\pi}{2}\right) = 0$.

    The tangent line at the point $\left(\frac{\pi}{2}, 4\right)$ is $y = 4$ and the normal line to the line $y = 4$ is $x = \frac{\pi}{2}$.

19. $g(t) = t\sin t \Rightarrow g'(t) = \sin t + t\cos t \Rightarrow g'\left(\frac{3\pi}{2}\right) = -1$.

    So, the tangent line at the point is $y + \frac{3\pi}{2} = -1\left(x - \frac{3\pi}{2}\right) \Rightarrow y = -x$.

    The normal line at the point is $y + \frac{3\pi}{2} = 1\left(x - \frac{3\pi}{2}\right) \Rightarrow y = x - 3\pi$.

21. $f(x) = x\sin x \Rightarrow f'(x) = \sin x + x\cos x = 0 \Rightarrow x = 0; y = 0$.

23. $f(x) = x\sin x \Rightarrow f'(x) = \sin x + x\cos x \Rightarrow f''(x) = \cos x + (\cos x - x\sin x)$.

    $\Rightarrow f^{(3)}(x) = -2\sin x - (\sin x + x\cos x)$.

    $\Rightarrow f^{(4)}(x) = -3\cos x - (\cos x - x\sin x) = -4\cos x + x\sin x$.

2.7 The Chain Rule

5. $f(x) = (4x^3 - x)^{10} \Rightarrow f'(x) = 10(4x^3 - x)^9(12x^2 - 1)$.

7. $g(\theta) = (\sin\theta + \cos\theta)^3 \Rightarrow g'(\theta) = 3(\sin\theta + \cos\theta)^2(\cos\theta - \sin\theta)$.

9. $f(x) = \left(x + \dfrac{1}{x}\right)^4 \Rightarrow f'(x) = 4\Big(x + \dfrac{1}{x})^3\Big(1 - \dfrac{1}{x^2}\Big)$.

11. $g(x) = \tan(5x) \Rightarrow g'(x) = 5\sec^2(5x)$.

13. $p(t) = \cos^3(t^2 + 3t + 1) \Rightarrow p'(t) = 3\cos^2(t^2 + 3t + 1) \cdot \big(-\sin(t^2 + 3t + 1)\big) \cdot (2t + 3)$.

15. $g(t) = \cos(t^2 + 3t) \cdot \sin(5t - 7) \Rightarrow g'(t) = -(2t+3)\sin(t^2+3t)\sin(5t-7) + 5\cos(t^2+3t)\cos(5t - 7)$.

17. $f(x) = (4x^3 - x)^{10} \Rightarrow f(0) = 0$. Also, $f'(x) = 10(4x^3 - x)^9(12x^2 - 1) \Rightarrow f'(0) = 0$.

    So the tangent line equation at the point $(0, 0)$ is $y = 0$ and the normal line equation is $x = 0$.

19. $g'(\theta) = 3(\sin\theta + \cos\theta)^2(\cos\theta - \sin\theta) \Rightarrow g\Big(\dfrac{\pi}{2}\Big) = 1 \Rightarrow g'\Big(\dfrac{\pi}{2}\Big) = -3$.

    So the tangent line equation at the point $\Big(\dfrac{\pi}{2}, 1\Big)$ is $y - 1 = -3\Big(x - \dfrac{\pi}{2}\Big)$.

    The normal line equation is $y - 1 = \dfrac{1}{3}\Big(x - \dfrac{\pi}{2}\Big)$.

2.8 Implicit Differentiation

13. $x^4 + y^2 + y = 7 \Rightarrow 4x^3 + 2y\dfrac{dy}{dx} + \dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{4x^3}{2y+1}.$

15. $\cos x + \sin y = 1 \Rightarrow -\sin x + \cos y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{\sin x}{\cos y}.$

17. $\dfrac{y}{x} = 10 \Rightarrow \dfrac{y'x - y}{x^2} = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{y}{x}.$

19. $x^2 \tan y = 50 \Rightarrow 2x\tan y + x^2\sec^2 y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{2x\tan y}{x^2\sec^2 y} = -\dfrac{2\sin y\cos y}{x}.$

21. $(y^2 + 2y - x)^2 = 200 \Rightarrow 2(y^2 + 2y - x)(2yy' + 2y' - 1) = 0.$

    $\Rightarrow y'(2y+2)(y^2 + 2y - x) = y^2 + 2y - x \Rightarrow y' = \dfrac{1}{2y+2}.$

23. $\dfrac{\sin x + y}{\cos y + x} = 1 \Rightarrow \dfrac{(\cos x + y')(\cos y + x) - (\sin x + y)(1 - \sin yy')}{(\cos y + x)^2} = 0.$

    $\Rightarrow y' = \dfrac{\sin x + y - \cos x(\cos y + x)}{\cos y + x + \sin y(\sin x + y)}$

25. $x^{0.4} + y^{0.4} = 1.$

    a) $0.4x^{-0.6} + 0.4y^{-0.6} \cdot y' = 0 \Rightarrow y' = -\dfrac{x^{-0.6}}{y^{-0.6}} = -\dfrac{y^{0.6}}{x^{0.6}}.$

    At $(1, 0), y' = 0$, so the tangent line is $y - 0 = 0(x - 1) \Rightarrow y = 0.$

    b) $y' = -1.859 \Rightarrow y - 0.281 = -1.859(x - 0.1) \Rightarrow y = -1.859(x - 0.1) + 0.281.$

27. $(x^2 + y^2 - 4)^3 = 108y^2.$

    a) $3(x^2 + y^2 - 4)^2\left(2x + 2y\dfrac{dy}{dx}\right) = 216y\dfrac{dy}{dx} \Rightarrow 3(0 + 16 - 4)^2\left(8\dfrac{dy}{dx}\right) = 864\dfrac{dy}{dx}$, at $(0, 4), y' = 0.$

    $\Rightarrow$ The tangent line is $y - 4 = 0(x - 0) \Rightarrow y = 4.$

    b) At $(2, -\sqrt[4]{108}), y' = 0.93.$ Then, the equation of the tangent line at the point is $y + \sqrt[4]{108} = 0.93(x - 2).$

29. $(x-2)^2+(y-3)^2=9$.

a) $2(x-2)+2(y-3)\dfrac{dy}{dx}=0 \Rightarrow 2\Big(\dfrac{7}{2}-2\Big)+2\Big(\dfrac{6+3\sqrt{3}}{2}-3\Big)\dfrac{dy}{dx}=0.$

At $\left(\dfrac{7}{2},\dfrac{6+3\sqrt{3}}{2}\right) \Rightarrow \dfrac{dy}{dx}=-\dfrac{1}{\sqrt{3}}$. Then, the equation of the tangent line at the point is $y-\dfrac{6+3\sqrt{3}}{2}=-\dfrac{1}{\sqrt{3}}\Big(x-\dfrac{7}{2}\Big)$.

b) $2\Big(\dfrac{4+3\sqrt{3}}{2}-2\Big)+2(\dfrac{3}{2}-3)\dfrac{dy}{dx}=0$, At $\left(\dfrac{4+3\sqrt{3}}{2},\dfrac{3}{2}\right) \Rightarrow \dfrac{dy}{dx}=\sqrt{3}$.

Then, the equation of the tangent line at the point is
$\Rightarrow y-\dfrac{3}{2}=\sqrt{3}\Big(x-\dfrac{4+3\sqrt{3}}{2}\Big)$.

31. $x^{0.4}+y^{0.4}=1 \Rightarrow 0.4x^{-0.6}+0.4y^{-0.6}\cdot y'=0 \Rightarrow y'=-\dfrac{y^{0.6}}{x^{0.6}}$.

$$y''=-\frac{0.6y^{-0.4}y'x^{0.6}-0.6y^{0.6}x^{-0.4}}{x^{1.2}}=\frac{3y^{0.6}}{5x^{1.6}}+\frac{3}{5yx^{1.2}}.$$

2.9 Derivatives of Exponential and Logarithmic Functions

9. $y = f(x) = 5x + 10 \Rightarrow x = 5y + 10 \Rightarrow y = \frac{1}{5}x - 2 \Rightarrow f^{-1}(x) = \frac{1}{5}x - 2$. Note that we don't need to find the inverse in this problem. Just want to show how to find one.

   $f'(x) = 5 \Rightarrow (f^{-1})'(20) = \frac{1}{f'(2)} = \frac{1}{5}$.

11. $f(x) = \sin 2x \Rightarrow f'(x) = 2\cos 2x \Rightarrow (f^{-1})'(\sqrt{3}/2) = \frac{1}{f'\left(\frac{\pi}{6}\right)} = \frac{1}{2\cos(\pi/3)} = \frac{1}{2 \cdot \frac{1}{2}} = 1$.

13. $f(x) = \frac{1}{1+x^2} \Rightarrow f'(x) = -\frac{2x}{(1+x^2)^2} \Rightarrow (f^{-1})'(1/2) = \frac{1}{f'(1)} = \frac{1}{-2/4} = -2$.

15. $f(x) = \ln(\cos x) \Rightarrow f'(x) = -\frac{\sin x}{\cos x} = -\tan x$.

17. $f(x) = 2\ln(x) \Rightarrow f'(x) = \frac{2}{x}$.

19. $g(t) = 5^{\cos t} \Rightarrow g'(t) = -\ln 5 \cdot 5^{\cos t} \cdot \sin t$.

21. $g(t) = 15^2 \Rightarrow g'(t) = 0$.

23. $m(w) = \frac{3^w + 1}{2^w} \Rightarrow m'(w) = \frac{\ln 3 \cdot 3^w \cdot 2^w - \ln 2 \cdot 2^w(3^w + 1)}{2^{2w}}$.

25. $h(t) = \frac{2^t + 3}{3^t + 2} \Rightarrow h'(t) = \frac{\ln 2 \cdot 2^t(3^t + 2) - \ln 3 \cdot 3^t(2^t + 3)}{(3^t + 2)^2}$.

27. a) $\frac{d}{dx}(\ln(x^k)) = kx^{k-1}\frac{1}{x^k} = \frac{k}{x}$.

    b) $\frac{d}{dx}(\ln(x^k)) = \frac{d}{dx}(k\ln(x)) = \frac{k}{x}$.

29. $y = (2x)^{x^2} \Rightarrow y(1) = 2$. Taking natural log both sides, simplifying, and then implicit differentiating gives $\ln(y) = x^2\ln(2x) \Rightarrow \frac{y'}{y} = 2x\ln(2x) + \frac{2x^2}{2x}$.

    $\Rightarrow y' = y(2x\ln(2x) + x) \Rightarrow y'(1) = 2(2\ln(2) + 1) = 4\ln(2) + 2$.

    The equation of the tangent line is $y - 2 = (4\ln(2) + 2)(x - 1)$.

31. $y = x^{\sin x+2} \Rightarrow y\left(\frac{\pi}{2}\right) = \frac{(\pi)^3}{8}$. Taking natural log both sides and simplifying gives $\ln(y) = (\sin x + 2)\ln(x)$.

Then implicit differentiating both sides gives $\frac{y'}{y} = \cos x \ln(x) + \frac{\sin x + 2}{x} \Rightarrow y' = y\left(\cos x \ln(x) + \frac{\sin x + 2}{x}\right) \Rightarrow y'\left(\frac{\pi}{2}\right) = \frac{(\pi)^3}{8} \cdot \frac{6}{\pi} = \frac{3(\pi)^2}{4}$.

The equation of the tangent line is $y - \frac{(\pi)^3}{8} = \frac{3(\pi)^2}{4}\left(x - \frac{\pi}{2}\right)$.

33. $y = \frac{(x+1)(x+2)}{(x+3)(x+4)} \Rightarrow y(0) = \frac{1}{6}$. Now, Taking natural log both sides and

simplifying gives

$\ln y = \ln(x+1) + \ln(x+2) - \ln(x+3) - \ln(x+4)$.

Then implicit differentiating both sides gives

$$\frac{y'}{y} = \frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x+3} - \frac{1}{x+4}$$
$$\Rightarrow y' = y\left(\frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x+3} - \frac{1}{x+4}\right) \Rightarrow y'\left(0\right) = \frac{1}{6} \cdot \left(\frac{1}{1} + \frac{1}{2} - \frac{1}{3} - \frac{1}{4}\right).$$

$\Rightarrow y'(0) = \frac{11}{72} \Rightarrow$ the equation of the tangent line is $y - \frac{1}{6} = \frac{11}{72}(x - 0)$
$\Rightarrow y = \frac{11}{72}x + \frac{1}{6}$.

2.10 Derivatives of Inverse Trigonometric Functions

1. $h(t) = \sin^{-1}(2t) \Rightarrow h'(t) = \dfrac{1}{\sqrt{1-4t^2}} \cdot 2 \Rightarrow h'(t) = \dfrac{2}{\sqrt{1-4t^2}}$.

3. $g(x) = \tan^{-1}(2x) \Rightarrow g'(x) = \dfrac{1}{1+4x^2} \cdot 2 \Rightarrow g'(x) = \dfrac{2}{1+4x^2}$.

5. $g(t) = \sin t \cos^{-1} t \Rightarrow g'(t) = \cos t \cos^{-1} t - \dfrac{\sin t}{\sqrt{1-t^2}}$.

7. $g(x) = \tan^{-1}(\sqrt{x}) \Rightarrow g'(x) = \dfrac{1}{1+x} \cdot \dfrac{1}{2\sqrt{x}} \Rightarrow g'(x) = \dfrac{1}{2\sqrt{x}(1+x)}$.

9. $f(x) = \sin(\sin^{-1} x) \Rightarrow f'(x) = \cos(\sin^{-1} x) \cdot \dfrac{1}{\sqrt{1-x^2}}$
   $\Rightarrow f'(x) = \dfrac{\cos(\sin^{-1} x)}{\sqrt{1-x^2}}$. Note that from precalculus, we can simplify this to $f(x) = \sin(\sin^{-1} x) = x \Rightarrow f'(x) = 1$.

11. $f(x) = \tan^{-1}(\tan x)$.

    a) $f(x) = x \Rightarrow f'(x) = 1$.

    b) $f'(x) = \dfrac{1}{1+\tan^2 x} \cdot \sec^2 x = \dfrac{\sec^2 x}{\sec^2 x} = 1$.

13. $f(x) = \sin^{-1} x \Rightarrow f'(x) = \dfrac{1}{\sqrt{1-x^2}} \Rightarrow f\left(\dfrac{\sqrt{2}}{2}\right) = \dfrac{\pi}{4} \Rightarrow f'\left(\dfrac{\sqrt{2}}{2}\right) = \sqrt{2}$.

    So, the equation of the tangent line at $x = \dfrac{\sqrt{2}}{2}$ is $y - \dfrac{\pi}{4} = \sqrt{2}\left(x - \dfrac{\sqrt{2}}{2}\right)$.

Chapter 3

Graphical Behaviour of Functions

3.1 Extreme Values

7. Absolute maximum is the point C.
   Absolute minimum is the point A.
   Relative maximums are B and D.
   Relative minimum is E.

9. $f'(x) = 2x(\sqrt{6-x^2}) - \dfrac{x^3}{2\sqrt{6-x^2}} \Rightarrow f'(2) = 0.$

13. $f'(0)$ is not defined.

15. $f(x) = x^2 + x + 4 \Rightarrow f'(x) = 2x + 1 = 0 \Rightarrow x = -\dfrac{1}{2}$. $f\Big(-\dfrac{1}{2}\Big) = \dfrac{15}{4}; f(-1) = 4; f(2) = 10.$

    The absolute minimum is at $x = -\dfrac{1}{2}$ and the absolute maximum is at $x = 2$.

17. $f(x) = 3\sin x \Rightarrow f\Big(\dfrac{\pi}{4}\Big) = \dfrac{3}{\sqrt{2}}; f\Big(\dfrac{2\pi}{3}\Big) = \dfrac{3\sqrt{3}}{2}$

    $\Rightarrow f'(x) = 3\cos x = 0 \Rightarrow x = \dfrac{\pi}{2} \Rightarrow f\Big(\dfrac{\pi}{2}\Big) = 3$ (max). And $f\Big(\dfrac{\pi}{4}\Big) = \dfrac{3}{\sqrt{2}}$ is min.

19. $f(x) = x + \dfrac{3}{x} \Rightarrow f(1) = 4; f(5) = \dfrac{28}{5}.$

    $f'(x) = 1 - \dfrac{3}{x^2} = 0 \Rightarrow x = \sqrt{3}; -\sqrt{3} \Rightarrow f(\sqrt{3}) = 2.732; f(-\sqrt{3}) = -0.732$ (min). $f(5) = \dfrac{28}{5}$ is max.

21. $f(x) = e^x \cos x \Rightarrow f(0) = 1; f(\pi) = -e^\pi.$

    $f'(x) = e^x(\cos x - \sin x) = 0 \Rightarrow x = \dfrac{\pi}{4} \Rightarrow f\Big(\dfrac{\pi}{4}\Big) = 1.551$ (max). And $f(\pi) = -e^\pi$ is min.

23. $f(x) = \dfrac{\ln x}{x} \Rightarrow f(1) = 0; f(4) = 0.347.$

    $f'(x) = \dfrac{1 - \ln x}{x^2} = 0 \Rightarrow x = e$. Thus, $f(e) = 0.368$ (max). And $f(1) = 0$ is min.

3.3 Increasing and Decreasing Functions

15. $f(x) = x^3 + 3x^2 + 3$.

    a) $D = (-\infty, +\infty)$.

    b) $f'(x) = 3x^2 + 6x = 0 \Rightarrow x = 0, -2$. $f'(x)$ is defined on $(-\infty, +\infty)$. So critical numbers are $x = 0, -2$.

    c) Put these two x values on signs chart (or numbers line). Pick some test values in the intervals and plug them in the $f'(x)$ above to see if it is positive or negative. After doing that we get

       $f'(x) > 0$ on $(-\infty, -2)$ & $(0, +\infty)$, on which the function is increasing; and $f'(x) < 0$ on $(-2, 0)$, on which the the function is decreasing.

    d) $f'(x)$ changes from positive to negative at $x = -2$ so the relative maximum is at $x = -2$. $f'(x)$ changes from negative to positive at $x = 0$ so the relative minimum is at $x = 0$.

17. $f(x) = x^3 - 3x^2 + 3x - 1$.

    a) $D = (-\infty, +\infty)$.

    b) $f'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x - 1)^2 = 0 \Rightarrow x = 1$.
       $f'(x)$ is defined on $(-\infty, +\infty)$. So, a critical number is $x = 1$.

    c) We can see from b) that $f'(x) > 0\ \forall x$. So, the function is always increase for all values of x.

    d) No sign change so there is no maximum and minimum.

19. $f(x) = \dfrac{x^2 - 4}{x^2 - 1}$.

    a) $D = (-\infty, -1) \cup (-1, 1) \cup (1, +\infty)$.

    b) $f'(x) = \dfrac{6x}{(x^2 - 1)^2} = 0 \Rightarrow x = 0$. $f'(x)$ is undefined at $x = -1, 1$. Thus, the critical numbers are $x = -1, 0, 1$

    c) Put these three x values on signs chart (or numbers line). Pick some test values in the intervals and plug them in the $f'(x)$ above to see if it is positive or negative. After doing that we get

       $f'(x) < 0$ on $(-\infty, 0)$, on which the function is decreasing; and $f'(x) > 0$ on $(0, +\infty)$, on which the function is increasing.

    d) $f'(x)$ changes from negative to positive at $x = 0$ so the relative minimum is at $x = 0$.

21. $f(x) = \dfrac{(x-2)^{\frac{2}{3}}}{x}$.

a) $D = (-\infty, 0) \cup (0, +\infty)$.

b) $f'(x) = \dfrac{6-x}{3x^2(x-2)^{\frac{1}{3}}} = 0 \Rightarrow x = 6$. $f'(x)$ is undefined at $x = 0, 2$.

So the critical numbers are $x = 0, 2, 6$.

c) Put these three x values on signs chart (or numbers line). Pick some test values in the intervals and plug them in the $f'(x)$ above to see if it is positive or negative. After doing that we get

$f'(x) < 0$ on $(-\infty, 2)$ & $(6, +\infty)$, on which the function is decreasing; and $f'(x) > 0$ on $(2, 6)$, on which the function is increasing.

d) $f'(x)$ changes from negative to positive at $x = 2$ so the relative minimum is at $x = 2$.

$f'(x)$ changes from positive to negative at $x = 6$ so the relative maximum is at $x = 6$.

23. $f(x) = x^5 - 5x$.

a) $D = (-\infty, +\infty)$.

b) $f'(x) = 5x^4 - 5 = 5(x-1)(x+1)(x^2+1) = 0 \Rightarrow x = -1, 1$.

$f'(x)$ is defined $\forall x$. So the critical points are at $x = -1, 1$.

c) Put these two x values on signs chart (or numbers line). Pick some test values in the intervals and plug them in the $f'(x)$ above to see if it is positive or negative. After doing that we get

$f'(x) > 0$ on $(-\infty, -1)$ & $(1, +\infty)$, on which the function is increasing; and $f'(x) < 0$ on $(-1, 1)$, on which the function is decreasing.

d) $f'(x)$ changes from negative to positive at $x = 1$ so the relative minimum is at $x = 1$.

$f'(x)$ changes from positive to negative at $x = -1$ so the relative maximum is at $x = -1$.

3.4 Concavity and Second Derivative

17. $f(x) = -x^2 - 5x + 7$.

   a) $f'(x) = -2x - 5 = 0 \Rightarrow x = -\frac{5}{2}$.
   $f''(x) = -2 < 0 \ \forall x$, thus there is no inflection point.

   b) $f''(x) = -2 < 0 \ \forall x$, thus $f(x)$ is concave down everywhere.

19. $f(x) = 2x^3 - 3x^2 + 9x + 5$.

   a) $f'(x) = 6x^2 - 6x + 9 = 3(2x^2 - 2x + 3) > 0 \ \forall x$.
   $f''(x) = 12x - 6 = 0 \Rightarrow x = \frac{1}{2}$. $f''(x)$ is defined everywhere. Thus the possible inflection point is at $x = \frac{1}{2}$.

   b) Put this x value on signs chart (or numbers line). Pick some test values in the intervals and plug them in the $f''(x)$ above to see if it is positive or negative. After doing that we get

   $f''(x) < 0$ (concave down) on $\left(-\infty, \frac{1}{2}\right)$ and $f''(x) > 0$ (concave up) on $\left(\frac{1}{2}, +\infty\right)$. Note that since the concavity changes direction at $x = \frac{1}{2}$, we can say that the inflection point is at $x = \frac{1}{2}$.

21. $f(x) = -3x^4 + 8x^3 + 6x^2 - 24x + 2$.

   a) $f'(x) = -12x^3 + 24x^2 + 12x - 24 = -12(x+1)(x-1)(x-2)$.

   $f''(x) = -36x^2 + 48x + 12 = 0 \Rightarrow x = \frac{2-\sqrt{7}}{3}, \frac{2+\sqrt{7}}{3}$. $f''(x)$ is defined everywhere. Thus, the possible inflection points are $x = \frac{2-\sqrt{7}}{3}, \frac{2+\sqrt{7}}{3}$.

   b) Put these x values on signs chart (or numbers line). Pick some test values in the intervals and plug them in the $f''(x)$ above to see if it is positive or negative. After doing that we get

   $f''(x) < 0$ (concave down) on $\left(-\infty, \frac{2-\sqrt{7}}{3}\right)$ & $\left(\frac{2+\sqrt{7}}{3}, +\infty\right)$, and $f''(x) > 0$ (concave up) on $\left(\frac{2-\sqrt{7}}{3}, \frac{2+\sqrt{7}}{3}\right)$. Note that since the concavity changes direction at these two points, we can say that the inflection points are at $x = \frac{2-\sqrt{7}}{3}, \frac{2+\sqrt{7}}{3}$.

23. $f(x) = \dfrac{1}{x^2+1}$.

a) $f'(x) = \dfrac{-2x}{(x^2+1)^2}$, and then $f''(x) = \dfrac{2(x^2-\frac{1}{3})}{(x^2+1)^3} = 0 \Rightarrow x = -\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$. $f''(x)$ is defined everywhere because $x^2+1 > 0\ \forall x$. Thus, the possible inflection points are $x = -\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$.

b) Put these x values on signs chart (or numbers line). Pick some test values in the intervals and plug them in the $f''(x)$ above to see if it is positive or negative. After doing that we get

$f''(x) > 0$ (concave up) on $\Big(-\infty, -\dfrac{1}{\sqrt{3}}\Big)$ & $\Big(\dfrac{1}{\sqrt{3}}, +\infty\Big)$ and $f''(x) < 0$ (concave down) on $\Big(-\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\Big)$. Note that since the concavity changes direction at these two points, we can say that the inflection points are at $x = -\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$.

25. $f(x) = \sin x + \cos x$.

a) $f'(x) = \cos x - \sin x$, and then $f''(x) = -\sin x - \cos x = 0 \Rightarrow x = -\dfrac{\pi}{4}, \dfrac{3\pi}{4}$. $f''(x)$ is defined everywhere. Thus, $f(x)$ has possible inflection points are at $x = -\dfrac{\pi}{4}, \dfrac{3\pi}{4}$.

b) Put these x values on signs chart (or numbers line). Pick some test values in the intervals and plug them in the $f''(x)$ above to see if it is positive or negative. After doing that we get

$f''(x) < 0$ (concave down) on $\Big(-\dfrac{\pi}{4}, \dfrac{3\pi}{4}\Big)$ and $f''(x) > 0$ (concave up) on $\Big(-\pi, -\dfrac{\pi}{4}\Big)$ & $\Big(\dfrac{3\pi}{4}, \pi\Big)$. Note that since the concavity changes direction at these two points, we can say that the inflection points are at $x = -\dfrac{\pi}{4}, \dfrac{3\pi}{4}$.

27. $f(x) = x^2 \ln x$.

a) $f'(x) = 2x\ln x + x$, and then $f''(x) = 2\ln x + 2 + 1 = 2\ln x + 3 = 0 \Rightarrow x = e^{-1.5}$. $f''(x)$ is defined on $(0, +\infty)$. Thus, the possible inflection point is at $x = e^{-1.5}$.

b) Put this x value on signs chart (or numbers line). Pick some test values in the intervals and plug them in the $f''(x)$ above to see if it is positive or negative. After doing that we get

$f''(x) < 0$ (concave down) on $(0, e^{-1.5})$ and $f''(x) > 0$ (concave up) on $(e^{-1.5}, +\infty)$. Note that since the concavity changes direction at these two points, we can say that the inflection points are at $x = e^{-1.5}$.

31. $f(x) = x^3 - x + 1$.

$f'(x) = 3x^2 - 1 = 0 \Rightarrow x = \dfrac{-1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$, and then $f''(x) = 6x \Rightarrow f''\Big(\dfrac{-1}{\sqrt{3}}\Big) < 0$. Thus, $f(x)$ has a local maximum at $x = \dfrac{-1}{\sqrt{3}}$. Also, $f''\Big(\dfrac{1}{\sqrt{3}}\Big) > 0$. Thus, $f(x)$ has a local minimum at $x = \dfrac{1}{\sqrt{3}}$.

35. $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$.

$f'(x) = 4x^3 - 12x^2 + 12x - 4 = 4(x-1)^3 = 0 \Rightarrow x = 1$, and then $f''(x) = 12x^2 - 24x + 12 = 12(x-1)^2 \geq 0 \Rightarrow f''(1) = 0$. Thus, $f(x)$ has a local minimum at $x = 1$.

37. $f(x) = \dfrac{x}{x^2 - 1}$.

$f'(x) = -\dfrac{x^2+1}{(x^2-1)^2} < 0 \ \forall x \neq -1, 1$. Thus, there is no maximum or minimum value.

39. $f(x) = x^2 e^x$.

$f'(x) = 2xe^x + x^2e^x = xe^x(x+2) = 0 \Rightarrow x = -2, 0$, and then $f''(x) = 2e^x + 2xe^x + 2xe^x + x^2e^x = e^x(x^2 + 4x + 2)$. $f''(0) > 0; f''(-2) < 0$. Thus, $f(x)$ has a local minimum at $x = 0$ and a local maximum at $x = -2$.

41. $f(x) = e^{-x^2}$.

$f'(x) = -2xe^{-x^2} = 0 \Rightarrow x = 0$, and then $f''(x) = -2e^{-x^2} + 4x^2e^{-x^2}$. $f''(0) < 0$. Thus, $f(x)$ has a local maximum at $x = 0$.

43. $f(x) = -x^2 - 5x + 7$.

$f'(x) = -2x - 5 = 0 \Rightarrow x = -\dfrac{5}{2}$, and then $f''(x) = -2 < 0 \ \forall x$. Thus, $f'(x)$ does not have maximum or minimum.

45. $f(x) = 2x^3 - 3x^2 + 9x + 5$.

$f'(x) = 6x^2 - 6x + 9 = 6\Big(x^2 - x + \dfrac{3}{2}\Big) > 0 \ \forall x$, and then $f''(x) = 12x - 6 = 0 \Rightarrow x = \dfrac{1}{2}$. $f'''(x) = 12 > 0$. Thus, $f'(x)$ has a minimum at $x = \dfrac{1}{2}$.

53. $f(x) = x^2 \ln x$.

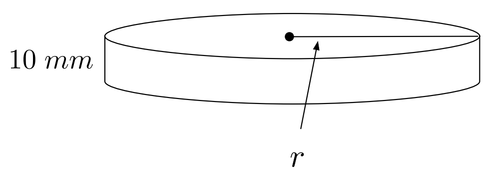
$f'(x) = 2x \ln x + x$, and then $f''(x) = 2\ln x + 3 = 0 \Rightarrow x = e^{-1.5}$. $f'''(x) = \dfrac{2}{x} \Rightarrow f'''(e^{-1.5}) > 0$. Thus, $f'(x)$ has a minimum at $x = e^{-1.5}$.

Chapter 4

Applications of the Derivatives

4.1 Related Rates Problems

3. Here, we want to find $\frac{dr}{dt}$.



$$V = \pi r^2 h \Rightarrow \frac{dV}{dr} = 2\pi r h \Rightarrow dr = \frac{dV}{2\pi r h}$$

a) $dr = \frac{5}{2\pi(1)0.1} = 7.96$ (mm)

b) $dr = \frac{5}{2\pi(10)0.1}1 = 0.796$ (mm)

c) $dr = \frac{5}{2\pi(100)0.1}1 = 0.0796$ (mm)

5. Diagram is similar to the one in question 4, but instead of East we have West.

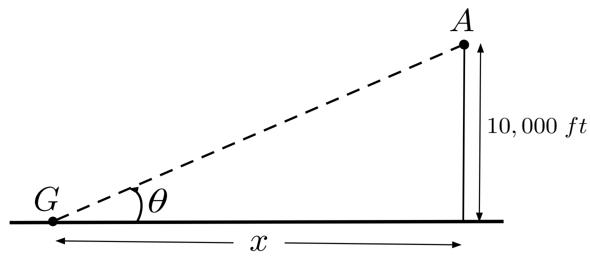
$A = \frac{1}{2}; B = \frac{1}{2} \Rightarrow C = \frac{1}{\sqrt{2}}.$

$\frac{dA}{dt} = -50; \frac{dC}{dt} = -80.$

$A^2 + B^2 = C^2 \Rightarrow 2A\frac{dA}{dt} + 2B\frac{dB}{dt} = 2C\frac{dC}{dt}.$

$\Rightarrow \frac{dB}{dt} = \frac{C\frac{dC}{dt} - A\frac{dA}{dt}}{B} = -63.14$ (mph).

7. Here, we want to find $\frac{d\theta}{dt}$.



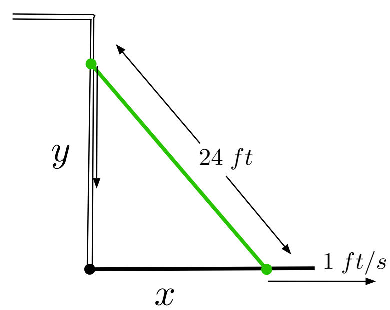
$$V = \pi r^2 h \Rightarrow \frac{dV}{dr} = 2\pi r h \Rightarrow dr = \frac{dV}{2\pi r h}$$

a) $dr = \frac{5}{2\pi(1)0.1} = 7.96$ (mm)

b) $dr = \frac{5}{2\pi(10)0.1}1 = 0.796$ (mm)

c) $dr = \frac{5}{2\pi(100)0.1}1 = 0.0796$ (mm)

9. Here, we want to find $\frac{dy}{dt}$.



$$x^2 + y^2 = 24^2 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

a) $y = \sqrt{24^2 - 1^2} = 23.98 \Rightarrow \frac{dy}{dt} = -\frac{1}{23.98} \cdot 1 = -0.0417$ (ft/s).

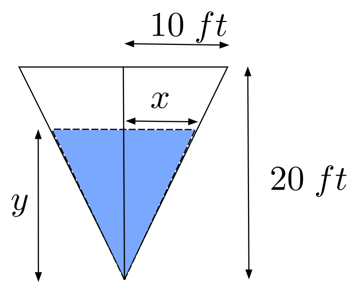
b) $y = \sqrt{24^2 - 10^2} = 21.82 \Rightarrow \frac{dy}{dt} = -\frac{10}{21.82} \cdot 1 = -0.458$ (ft/s).

c) $y = \sqrt{24^2 - 23^2} = 6.86 \Rightarrow \frac{dy}{dt} = -\frac{23}{6.86} \cdot 1 = -3.353$ (ft/s).

d) $\frac{dy}{dt} = \lim_{x \to 24} -\frac{x}{\sqrt{24^2 - x^2}} \cdot 1 = -\infty$. So it does not exist.

11. $\dfrac{x}{y} = \dfrac{1}{4} \Rightarrow y = 4x.$

$V = \dfrac{1}{3}\pi x^2 y = \dfrac{1}{3}\pi y \dfrac{y^2}{16} = \dfrac{1}{48}\pi y^3 \Rightarrow \dfrac{dV}{dt} = \dfrac{1}{16}\pi y^2 \dfrac{dy}{dt}.$



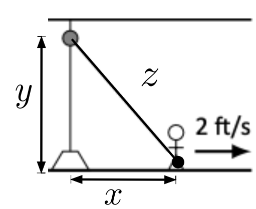
a) $\dfrac{dy}{dt} = \dfrac{16 \cdot 10}{\pi(1)^2} = 50.93$ (ft/min).

b) $\dfrac{dy}{dt} = \dfrac{16 \cdot 10}{\pi(10)^2} = 0.5093$ (ft/min).

c) $\dfrac{dy}{dt} = \dfrac{16 \cdot 10}{\pi(19)^2} = 0.141$ (ft/min).

$V = \dfrac{1}{3}\pi 5^2 \cdot 20 = 523.6 \Rightarrow t = \dfrac{523.6}{10} = 52.36$ (mins).

13. Solution here



a) $z^2 = x^2 + y^2 = 30^2 + 40^2 = 2500 \Rightarrow z = 50 \Rightarrow L = 30 + 50 = 80$ (ft).

b) $x = 50 \Rightarrow z = \sqrt{50^2 + 30^2} = 10\sqrt{34}.$

$30^2 + x^2 = z^2 \Rightarrow 0 + x\dfrac{dx}{dt} = z\dfrac{dz}{dt} \Rightarrow \dfrac{dz}{dt} = \dfrac{50 \cdot 2}{10\sqrt{34}} = 1.715.$

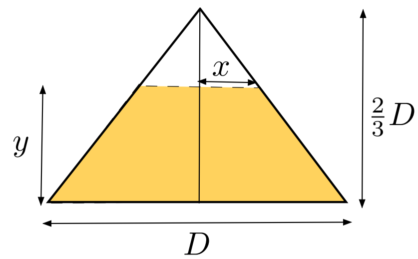
c) $x = 80 \Rightarrow z = \sqrt{80^2 + 30^2} = 10\sqrt{73}.$

$30^2 + x^2 = z^2 \Rightarrow 0 + x\dfrac{dx}{dt} = z\dfrac{dz}{dt} \Rightarrow \dfrac{dz}{dt} = \dfrac{80 \cdot 2}{10\sqrt{73}} = 1.873.$

d) $z = 80 \Rightarrow x = \sqrt{80^2 - 30^2} = 74.16.$

The man needs to walk $74.16 - 40 = 34.16$ (ft).

15. $h = \dfrac{2}{3}d = \dfrac{4}{3}r \Rightarrow r = \dfrac{3}{4}h.$



$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \frac{9}{16}h^3 = \frac{3}{16}\pi h^3.$$
$$\frac{dV}{dt} = \frac{9}{16}\pi h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = 0.00314 \text{ (ft/s)}.$$

4.2 Applied Optimization Problems

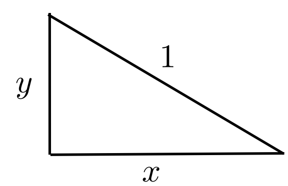
3. Let P be the product of two numbers x and y, such that $x + y = 100$. Here, we want to maximize P and find it value.

   $y = 100 - x \Rightarrow P = xy = x(100 - x) = -x^2 + 100x$. Then, $P'(x) = -2x + 100 = 0 \Rightarrow x = 50$. $P''(x) = -2 < 0 \Rightarrow P(x)$ has a maximum at $x = 50 \Rightarrow y = 50$.

5. Let S be the sum of two numbers x and y, such that $x \cdot y = 500$. Here, we want to maximize S and find it value.

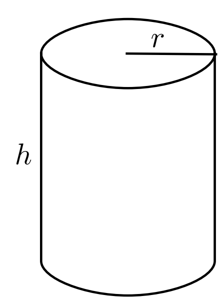
   $y = \dfrac{500}{x} \Rightarrow S = x+y = x+\dfrac{500}{x}$. Then, $S'(x) = 1-\dfrac{500}{x^2} = 0 \Rightarrow x^2 = 500 \Rightarrow x = -10\sqrt{5}, 10\sqrt{5}$.
   $S''(x) = \dfrac{1000}{x^3} \Rightarrow S''(-10\sqrt{5}) < 0$ and $S''(10\sqrt{5}) > 0$. Thus, $S(x)$ has maximum at $x = -10\sqrt{5}$.

7. Let A be the area of the right triangle below. Here, we want to maximize A and find it value.

   $x^2 + y^2 = 1 \Rightarrow y = \sqrt{1 - x^2}$.
   $A = \dfrac{1}{2}xy = \dfrac{1}{2}x\sqrt{1 - x^2}$. Then, $A'(x) = \dfrac{1}{2}\sqrt{1 - x^2} - \dfrac{x^2}{2\sqrt{1 - x^2}} = 0 \Rightarrow x = \dfrac{1}{\sqrt{2}}$.
   $A'(x)$ changes the sign from positive to negative at $x = \dfrac{1}{\sqrt{2}}$. Thus, the maximum area happens if $x = y = \dfrac{1}{\sqrt{2}}$.

9. Here, we want to find r, h.

$V = \pi r^2 h \Rightarrow h = \dfrac{V}{\pi r^2}$. Also, $A = 2\pi rh + 2\pi r^2 = 2\pi r \dfrac{V}{\pi r^2} + 2\pi r^2 = \dfrac{710}{r} + 2\pi r^2$.

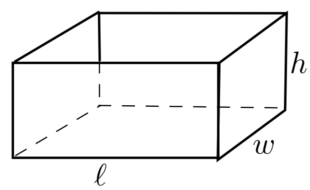
a) Minimize the amount of material M needed to produce the can.

$A'(r) = -\dfrac{710}{r^2} + 4\pi r = 0 \Rightarrow r = 3.837 \Rightarrow h = 7.596$. $A'(r)$ changes the sign from negative to positive at $r = 3.837, h = 7.596$. Thus, the minimum happens if $r = 3.837, A = M = 277.545$.

b) Minimize the costs C needed to produce the can.

Let P be price unit for each cm^2 of the material. $C = A \cdot P$. Thus, the standard can is produced to minimize the cost.

11. Let V be the volume of the box below. Here, we want to maximize V when $w = h$.

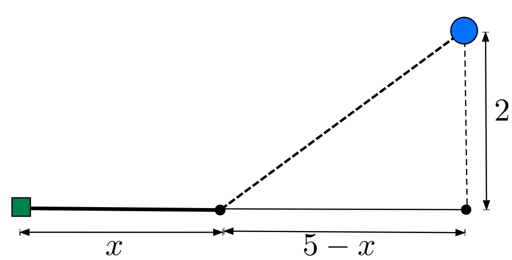


$\ell + 2w + 2h = 108 \Rightarrow \ell + 4w = 108 \Rightarrow \ell = 108 - 4w$.

We have $V = \ell \cdot w \cdot h = (108 - 4w)ww = 108w^2 - 4w^3$. Then, $V'(w) = 216w - 12w^2 = 0 \Rightarrow w = 18 \Rightarrow h = 18 \Rightarrow \ell = 36$.

The sign of $V'(w)$ changes from positive to negative at $w = 18$. Thus, V has a maximum at $w = 18$.

13. Let C be the cost to lay to power line underground. Here, we want find x to minimize C.



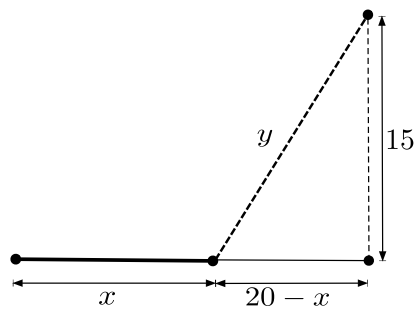
$C = 50x + 80\sqrt{4 + (5 - x)^2} = 50x + 80\sqrt{x^2 - 10x + 29}$.

$C'(x) = 50 + \dfrac{40(2x - 10)}{\sqrt{x^2 - 10x + 29}} = 0 \Rightarrow x = 6.6$ (invalid because we need $x < 5$, and $x = 3.4$. So the cost is minimum at $x = 3.4$.

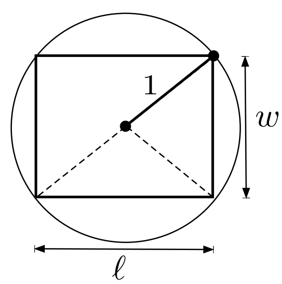
15. Let T be the time the dog takes to get the stick. Here, we want to find x to minimize T.

$T = \dfrac{x}{22} + \dfrac{y}{1.5} = \dfrac{x}{22} + \dfrac{\sqrt{15^2 + (20-x)^2}}{1.5}$. Then, we have

$T'(x) = \dfrac{1}{22} + \dfrac{2x - 40}{3\sqrt{x^2 - 40x + 625}} = 0 \Rightarrow x = 19.4$. So, the dog needs to run 19.4 feet along the shore to minimize the time.



17. Let A be the area of the rectangle inscribed inside the unit circle ($r = 1$), depicted in the following diagram. Here, we want to find w and ℓ to maximize A.



$w^2 + \ell^2 = (2r)^2 = 4 \Rightarrow \ell = \sqrt{4 - w^2}$.

$A = \dfrac{1}{2}\ell \cdot w = \dfrac{1}{2}w(\sqrt{4 - w^2})$. Then, $A'(w) = \dfrac{1}{2}\left(\sqrt{4 - w^2} - \dfrac{w^2}{\sqrt{4 - w^2}}\right) = 0 \Rightarrow w = \sqrt{2} \Rightarrow \ell = \sqrt{2} \Rightarrow A = 2$.

4.3 Economics and Business Applications

1. $x = 500 - 10p \Rightarrow x' = -10$.

   a) $E(p) = -\dfrac{p}{50 - p}$.

   b) $E(30) = -1.5$. Since $|E| = 1.5 > 1$, the demand is elastic.

   c) The percentage change in demand is $(-1.5)(4.5\%) = -6.75\%$.

3. $x^2 + p^2 = 4 \Rightarrow 2xx' + 2p = 0 \Rightarrow x' = -\dfrac{p}{x}$.

   a) $E(p) = -\dfrac{p^2}{4 - p^2}$.

   b) If $x = 1$, then $p = \sqrt{3}$. So, $E(\sqrt{3}) = -3$. Since $|E| = 3 > 1$, the demand is elastic.

   c) Now, we can solve for E in term of x from (a). Here, we get $E = \dfrac{x^2 - 4}{x^2}$. So, elastic on $0 < x < 1.42$ and inelastic on $1.42 < x < 2$.

5. $p = \left(100 - \dfrac{x}{10}\right)^2 \Rightarrow x = 1000 - 10\sqrt{p} \Rightarrow x' = -\dfrac{5}{\sqrt{p}}$.

   a) $E(p) = -\dfrac{500}{\sqrt{100}(1000 - 10\sqrt{p})}$.

   b) Currently, at $p = 100$, $|E| = 0.056 < 1$, the demand is inelastic. Therefore, the price per unit should be increased in order to increased the revenue.

7. $\ln x - 2\ln p + 0.02p = 7 \Rightarrow \dfrac{x'}{x} = \dfrac{2}{p} - 0.02$.

   a) Here, we have $E = \dfrac{100 - p}{50}$. Currently, at $p = 200$, $|E| = 2 > 1$, the demand is elastic. Therefore, we should increase the revenue.

   b) When $|E| = 1 \Rightarrow 100 - p = 50 \Rightarrow p = 150$.

9. $V(t) = 100(60 + t^2) \Rightarrow V'(t) = 100(0 + 2t) = 200t$.

   $r = \dfrac{V'}{V} \Rightarrow 0.0625 = \dfrac{2t}{60 + t^2} \Rightarrow t = 2,\ 30$. Now, we check $r'(t) = \dfrac{2(60 - t^2)}{(60 + t^2)^2}$. Since $r'(2) > 0$ and $r'(30) < 0$, the present value is a minimum at $t = 2$ (good to buy), and is maximum at $t = 30$, when the asset should be sold.

11. $V(t) = 250(5 + 0.2t)^{3/2} \Rightarrow V'(t) = 75(5 + 0.2t)^{1/2}$.

$r = \dfrac{V'}{V} \Rightarrow 0.04 = \dfrac{75(5 + 0.2t)^{1/2}}{250(5 + 0.2t)^{3/2}} = \dfrac{3}{10(5 + 0.2t)} \Rightarrow t = 12.5$. Sell in about 12.5 years. $V(12.5) = \$5134.90$.

13. $V(t) = 50te^{-0.10t} \Rightarrow V'(t) = 50e^{-0.10t}(1 - 0.10t)$.

a) $V'(t) = 0 \Rightarrow t_c = 10$ years. When $0 \leq t < 10, V' > 0$ and when $t > 10, V' < 0$. So, $V(10)$ is a local maximum. Because the domain of V is all real numbers, and there is only one critical number $t_c = 10$, this local maximum is the absolute maximum.

b) $r = \dfrac{V'(t)}{V(t)} \Rightarrow 0.025 = \dfrac{50e^{-0.10t}(1 - 0.10t)}{50te^{-0.10t}} = 1 - 0.10t \Rightarrow t = 8$ years.

15. $V(t) = 100(60t - t^2) \Rightarrow V'(t) = 100(60 - 2t) = 0 \Rightarrow t = 30$, that is, $t_c = 30$.

When $0 \leq t < 30, V' > 0$ and when $t > 30, V' < 0$. So, $V(30)$ is a local maximum. Because the domain of V is all real numbers, and there is only one critical number $t_c = 30$, this local maximum is the absolute maximum.

17. Here $C_0(x) = \dfrac{Q}{x} \cdot s = \dfrac{10,000}{x} \cdot (4.50) = \dfrac{45,000}{x}$. $C_s(x) = \dfrac{x}{2} \cdot r = \dfrac{x}{2} \cdot 5.76 = 2.88x$. The total cost function is then $C(x) = \dfrac{45,000}{x} + 2.88x$. $C'(x) = -\dfrac{45,000}{x^2} + 2.88 = 0 \Rightarrow x = 125$. Since $C''(x) = \dfrac{90,000}{x^3} > 0$ for all $x > 0$, $C(x)$ has a min. at $x = 125$. The number of times that the merchant should order is $\dfrac{10,000}{125} = 80$ in order minimize the total cost. The minimum total cost is $C(125) = \dfrac{45,000}{125} + 2.88(125) = 720$.

4.4 Linear Approximation and Differentials

7. $f(x) = x^2 \Rightarrow f'(x) = 2x$.

   Using linear approximation (or tangent line approximation), we take

   $a = 6 \Rightarrow f'(a) = 12 \Rightarrow f(a) = 36 \Rightarrow y - 36 = 12(x - 6)$.

   $\Rightarrow y = 12x - 36 \Rightarrow 5.93^2 \approx 12(5.93) - 36 = 35.16$.

   Or we can use differential form of linear approximation as follows. Here, $dx = 5.93 - 6 = -0.07$.

   $dy = 12\ dx = 12 \cdot (-0.07) = -0.84 \Rightarrow 5.93^2 \approx 6^2 - 0.84 = 35.16$.

9. $f(x) = x^3 \Rightarrow f'(x) = 3x^2$. Here, $dx = 6.8 - 7 = -0.2$.

   Take $a = 7 \Rightarrow dy = 3(7)^2\ dx = 147 \cdot (-0.2) = -29.4$

   $\Rightarrow 6.8^3 = 7^3 - 29.4 \approx 313.6$.

11. $f(x) = \sqrt{x} \Rightarrow f'(x) = \dfrac{1}{2\sqrt{x}}$. Here, $dx = 24 - 25 = -1$.

    Take $a = 25 \Rightarrow dy = \dfrac{1}{2 \cdot 5}\ dx = -0.1 \Rightarrow \sqrt{24} \approx \sqrt{25} - 0.1 = 4.9$.

13. $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}} \Rightarrow f'(x) = \dfrac{1}{3}x^{-\frac{2}{3}}$. Here, $dx = 8.5 - 8 = 0.5$.

    Take $a = 8 \Rightarrow dy = \dfrac{1}{3}8^{-\frac{2}{3}}\ dx = \dfrac{1}{12} \Rightarrow 8.5^{\frac{1}{3}} \approx \sqrt[3]{8} + \dfrac{1}{12} \approx 2.083$.

15. $f(x) = \cos x \Rightarrow f'(x) = -\sin x$.

    Since $\dfrac{\pi}{2} \approx 1.571$, we take $a = \dfrac{\pi}{2} \Rightarrow dx = 1.5 - \dfrac{\pi}{2} = -0.071$

    $\Rightarrow dy = -\sin\dfrac{\pi}{2}\ dx = (-1) \cdot (-0.071) = 0.071$

    $\Rightarrow \cos 1.5 \approx \cos\dfrac{\pi}{2} + 0.071 = 0.071$.

    Note: These problems above can be done using the tangent line approximation as done in #7 as well. The answers will be the same.

17. $y = x^2 + 3x - 5 \Rightarrow dy = (2x + 3)\ dx.$

19. $y = \dfrac{1}{4x^2} \Rightarrow dy = -\dfrac{1}{2}x^{-3}\ dx.$

21. $y = x^2 e^{3x} \Rightarrow dy = (2x \cdot e^{3x} + x^2 \cdot e^{3x} \cdot 3)dx = x(2 + 3x)e^{3x}\ dx.$

23. $y = \dfrac{2x}{\tan x + 1} \Rightarrow dy = \dfrac{2(\tan x + 1) + 2x \sec^2 x}{(\tan x + 1)^2}\ dx.$

25. $y = e^x \sin x \Rightarrow dy = (e^x \sin x + e^x \cos x)\ dx = (\sin x + \cos x)e^x\ dx.$

27. $y = \dfrac{x+1}{x+2} \Rightarrow dy = \dfrac{1 \cdot (x+2) - (x+1) \cdot 1}{(x+2)^2}\ dx = \dfrac{1}{(x+2)^2}\ dx.$

29. $y = x \ln x - x \Rightarrow dy = (1 \cdot \ln x + x \cdot \dfrac{1}{x} - 1)\ dx = \ln x\ dx.$

4.5 L'Hopital's Rule

11. $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{\sin x - \cos x}{\cos 2x} = \lim\limits_{x\to\frac{\pi}{4}} \dfrac{\cos x + \sin x}{-2\sin 2x} = -\dfrac{1}{\sqrt{2}}.$

13. $\lim\limits_{x\to 0} \dfrac{\sin 2x}{x+2} = \dfrac{\sin 0}{0+2} = 0.$

15 $\lim\limits_{x\to 0} \dfrac{\sin ax}{\sin bx} = \lim\limits_{x\to 0} \dfrac{a\cos ax}{b\cos bx} = \dfrac{a}{b}.$

17. $\lim\limits_{x\to 0^+} \dfrac{e^x - x - 1}{x^2} = \lim\limits_{x\to 0^+} \dfrac{e^x - 1}{2x} = \lim\limits_{x\to 0^+} \dfrac{e^x}{2} = \dfrac{1}{2}.$

21. $\lim\limits_{x\to +\infty} \dfrac{e^x}{\sqrt{x}} = \lim\limits_{x\to +\infty} 2e^x\sqrt{x} = +\infty.$

25. $\lim\limits_{x\to -2} \dfrac{x^3+4x^2+4x}{x^3+7x^2+16x+12} = \lim\limits_{x\to -2} \dfrac{3x^2+8x+4}{3x^2+14x+16} = \lim\limits_{x\to -2} \dfrac{6x+8}{6x+14} = -2.$

27. $\lim\limits_{x\to \infty} \dfrac{\ln x^2}{x} = \lim\limits_{x\to \infty} \dfrac{2}{x} = 0.$

29. $\lim\limits_{x\to 0^+} x\cdot \ln x = \lim\limits_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x}} = \lim\limits_{x\to 0^+} \dfrac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim\limits_{x\to 0^+} (-x) = 0.$

35. $\lim\limits_{x\to 0^+} \dfrac{1}{x^2} e^{-\frac{1}{x}} = \lim\limits_{x\to 0^+} \dfrac{\frac{1}{x^2}}{e^{\frac{1}{x}}} = \lim\limits_{x\to 0^+} \dfrac{-2x^{-3}}{-e^{\frac{1}{x}}x^{-2}} = \lim\limits_{x\to 0^+} \dfrac{-2x^{-2}}{-e^{\frac{1}{x}}x^{-2}} = \lim\limits_{x\to 0^+} \dfrac{2}{e^{\frac{1}{x}}} = 0.$

37. Let $y = \lim\limits_{x\to 0^+} (2x)^x \Rightarrow \ln y = \lim\limits_{x\to 0^+} \ln(2x)^x = \lim\limits_{x\to 0^+} x\ln 2x = \lim\limits_{x\to 0^+} \dfrac{\ln 2x}{\frac{1}{x}}$

$= \lim\limits_{x\to 0^+} \dfrac{\frac{2}{x}}{-\frac{1}{x^2}} = \lim\limits_{x\to 0^+} (-2x) = 0 \Rightarrow \lim\limits_{x\to 0^+} (2x)^x = e^0 = 1.$

Or: We write $\lim\limits_{x\to 0^+} (2x)^x = e^{\lim\limits_{x\to 0^+} \ln(2x)^x}$. Now, we compute

$\lim\limits_{x\to 0^+} x\ln 2x = \lim\limits_{x\to 0^+} \dfrac{\ln 2x}{\frac{1}{x}} = \lim\limits_{x\to 0^+} \dfrac{\frac{2}{x}}{-\frac{1}{x^2}} = \lim\limits_{x\to 0^+} (-2x) = 0 \Rightarrow \lim\limits_{x\to 0^+} (2x)^x = e^0 = 1.$

41. Let $y = \lim_{x\to\infty} x^{\frac{1}{x}} \Rightarrow \ln y = \lim_{x\to\infty} \ln x^{\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{x}\ln x = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$

$\Rightarrow \lim_{x\to\infty} x^{\frac{1}{x}} = e^0 = 1.$

43. Let $y = \lim_{x\to 1^+} (\ln x^{1-x}) \Rightarrow \ln y = \lim_{x\to 1^+} \ln(\ln x^{1-x}) = \lim_{x\to 1^+} (1-x)\ln(\ln x)$

$= \lim_{x\to 1^+} \frac{\ln(\ln x)}{\frac{1}{1-x}} = \lim_{x\to 1^+} \frac{\frac{1}{x\ln x}}{\frac{1}{(1-x)^2}} = \lim_{x\to 1^+} \frac{(1-x)^2}{x\ln x} = \lim_{x\to 1^+} \frac{-2(1-x)}{\ln x + 1} = 0$

$\Rightarrow \lim_{x\to 1^+} (\ln x^{1-x}) = e^0 = 1.$

45. We write $\lim_{x\to\infty} (1+x^2)^{\frac{1}{x}} = e^{\lim_{x\to\infty} \ln[(1+x^2)^{\frac{1}{x}}]}$. Now, we compute

$\lim_{x\to\infty} \ln[(1+x^2)^{\frac{1}{x}}] = \lim_{x\to\infty} \frac{1}{x}\ln(1+x^2) = \lim_{x\to\infty} \frac{\frac{2x}{1+x^2}}{1} = \lim_{x\to\infty} \frac{2}{2x} = 0.$

So, we get $\lim_{x\to\infty} (1+x^2)^{\frac{1}{x}} = e^0 = 1.$

49. $\lim_{x\to 3^+} \frac{5}{x^2-9} - \frac{x}{x-3} = \lim_{x\to 3^+} \frac{5-x(x+3)}{(x-3)(x+3)} = \lim_{x\to 3^+} \frac{-x^2-3x+5}{(x-3)(x+3)} = -\infty.$

51. $\lim_{x\to\infty} \frac{(\ln x)^3}{x} = \lim_{x\to\infty} \frac{3(\ln x)^2}{x} = \lim_{x\to\infty} \frac{6\ln x}{x} = \lim_{x\to\infty} \frac{\frac{6}{x}}{1} = 0.$

4.6 Antiderivatives and Indefinite Integration

9. $\int x^8\,dx = \frac{1}{9}x^9 + C.$

11. $\int dt = t + C.$

13. $\int \frac{1}{3t^2}\,dt = -\frac{1}{3t} + C.$

15. $\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x} + C.$

17. $\int \sin\theta\,d\theta = -\cos\theta + C.$

19. $\int 5e^{\theta}\,d\theta = 5e^{\theta} + C.$

21. $\int \frac{5^t}{2}\,dt = \frac{5^t}{2\ln 5} + C.$

23. $\int (t^2+3)(t^3-2t)\,dt = \int (t^5+t^3-6t)\,dt = \frac{1}{6}t^6 + \frac{1}{4}t^4 - 3t^2 + C.$

25. $\int e^{\pi}\,dx = e^{\pi}x + C.$

29. $f(x) = \int 5e^x\,dx = 5e^x + C.$

    $f(0) = 10 \Rightarrow 5 + C = 10 \Rightarrow C = 5 \Rightarrow f(x) = 5e^x + 5.$

31. $f(x) = \int \sec^2 x\,dx = \tan x + C.$

    $f\left(\frac{\pi}{4}\right) = 5 \Rightarrow 1 + C = 5 \Rightarrow C = 4 \Rightarrow f(x) = \tan x + 4.$

33. $f'(x) = \int 5\,dx = 5x + C.$

$f'(0) = 7 \Rightarrow C = 7 \Rightarrow f'(x) = 5x + 7.$

$f(x) = \int (5x + 7)\,dx = \frac{5}{2}x^2 + 7x + D.$

$f(0) = 3 \Rightarrow D = 3 \Rightarrow f(x) = \frac{5}{2}x^2 + 7x + 3.$

35. $f'(x) = \int 5e^x\,dx = 5e^x + C.$

$f'(0) = 3 \Rightarrow C = -2 \Rightarrow f'(x) = 5e^x - 2.$

$f(x) = \int (5e^x - 2)\,dx = 5e^x - 2x + D.$

$f(0) = 5 \Rightarrow D = 0 \Rightarrow f(x) = 5e^x - 2x.$

37. $f'(x) = \int (24x^2 + 2^x - \cos x)\,dx = 8x^3 + \frac{2^x}{\ln 2} - \sin x + C.$

$f'(0) = 5 \Rightarrow C = -\frac{1}{\ln 2} \Rightarrow f'(x) = 8x^3 + \frac{2^x}{\ln 2} - \sin x - \frac{1}{\ln 2}.$

$f(x) = \int \left(8x^3 + \frac{2^x}{\ln 2} - \sin x - \frac{1}{\ln 2}\right)\,dx = 2x^4 + \frac{2^x}{(\ln 2)^2} + \cos x - \frac{x}{\ln 2} + D.$

$f(0) = 0 \Rightarrow D = -\frac{1}{(\ln 2)^2} - 1.$

$\Rightarrow f(x) = 2x^4 + 2^x(\ln 2)^2 + \cos x - x\ln 2 - \frac{1}{(\ln 2)^2} - 1.$