Mathematics 1274

SOLUTIONS TO SELECTED EXERCISES FROM MATH 1274 TEXTBOOK

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Minh Phuong Bui & Pichmony Anhaouy
Langara College
Mathematics and Statistics department
panhaouy@langara.ca

Contents

Chapter 1

Integration

1.1 Antiderivatives and Indefinite Integration

9. $\int x^8\,dx = \frac{x^9}{9} + C.$

11. $\int dt = t + C.$

13. $\int \frac{1}{3t^2}\,dt = \frac{-1}{3t} + C.$

15. $\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x} + C.$

17. $\int \sin\theta\,d\theta = -\cos\theta + C.$

19. $\int 5e^\theta\,d\theta = 5e^\theta + C.$

21. $\int \frac{5^t}{2}\,dt = \frac{5^t}{2\ln(5)} + C.$

23. $\int (t^2+3)(t^3-2t)\,dt = \int (t^5+t^3-6t)\,dt = \frac{t^6}{6} + \frac{t^4}{4} - 3t^2 + C.$

25. $\int e^\pi\,dx = e^\pi x + C.$

27. $\int \frac{1}{x}\,dx = \ln(|x|) + C.$

   (a) What is the domain?

   The function is $\ln(|x|)$ so we need x > 0. The domain is $(0,\infty)$.

   (b) $\frac{d}{dx}\Big[\ln(x)\Big] = \frac{1}{x}$

(c) What is the domain of $\ln(-x)$?

We need $-x > 0$ therefore $x < 0$. Domain of the function $\ln(-x)$ is $(-\infty, 0)$.

(d) Find $\frac{d}{dx}$ of $\ln(-x)$.

$$\frac{d}{dx}\Big[\ln(-x)\Big] = -\frac{1}{x}$$

(e) $\int \frac{1}{x}\,dx = \begin{cases} \ln(-x), & \text{if } x < 0 \\ \ln(x), & \text{if } x > 0 \end{cases}$

29. $f(x) = \int f'(x)\,dx = \int \sin(x)\,dx = -\cos x + C.$

$$f(0) = 2 \Rightarrow -\cos 0 + C = 2, \Rightarrow -1 + C = 2 \Rightarrow C = 3 \Rightarrow f(x) = -\cos x + 3.$$

31. $f(x) = \int f'(x)\,dx = \int \sec(x)^2\,dx = \tan x + C.$

$$f(\frac{\pi}{4}) = 5 \Rightarrow \tan\frac{\pi}{4} + C = 5, \Rightarrow 1 + C = 5 \Rightarrow C = 4 \Rightarrow f(x) = \tan x + 4.$$

33. $f'(x) = \int f"(x)\,dx = \int 5\,dx = 5x + C.$

$$f'(0) = 7 \Rightarrow 5 * 0 + C = 7, \Rightarrow 0 + C = 7 \Rightarrow C = 7 \Rightarrow f'(x) = 5x + 7.$$

$$f(x) = \int f'(x)\,dx = \int (5x+7)\,dx = \frac{5x^2}{2} + 7x + C.$$

$$f(0) = 3 \Rightarrow 0 + 0 + C = 3, \Rightarrow 0 + C = 3 \Rightarrow C = 3 \Rightarrow f(x) = \frac{5x^2}{2} + 7x + 3.$$

35. $f'(x) = \int f"(x)\,dx = \int 5e^x\,dx = 5e^x + C.$

$$f'(0) = 3 \Rightarrow 5 * e^0 + C = 3, \Rightarrow 5 + C = 3 \Rightarrow C = -2 \Rightarrow f'(x) = 5e^x - 2.$$

$$f(x) = \int f'(x)\,dx = \int (5^e x - 2)\,dx = 5e^x - 2x + C.$$

$$f(0) = 5 \Rightarrow 5 - 0 + C = 5, \Rightarrow 5 + C = 5 \Rightarrow C = 0 \Rightarrow f(x) = 5e^x - 2x.$$

1.2 The Definite Integral

5. (a) $\int_0^1 (-2x+4)\,dx = (1*2) + \frac{1}{2} * (1*2) = 2 + 1 = 3.$

(b) $\int_0^2 (-2x+4)\,dx = \frac{1}{2} * (2*4) = 4.$

(c) $\int_0^3 (-2x+4)\,dx = 4 - \frac{1}{2} * (1*2) = 3.$

(d) $\int_1^3 (-2x+4)\,dx = \int_1^2 (-2x+4)\,dx + \int_2^3 (-2x+4)\,dx = 0.$

(e) $\int_2^4 (-2x+4)\,dx = -\frac{1}{2} * (2*4) = -4.$

(f) $\int_0^1 (-6x+12)\,dx = 3\int_0^1 (-2x+4)\,dx = 3*3 = 9.$

9. (a) $\int_0^2 f(x)\,dx = \frac{\pi 2^2}{4} = \pi.$

(b) $\int_2^4 f(x)\,dx = \frac{\pi 2^2}{4} = \pi.$

(c) $\int_0^4 f(x)\,dx = \frac{\pi 2^2}{2} = 2\pi.$

(d) $\int_0^4 5f(x)\,dx = 5*2 = 10.$

19. $\int_0^3 (f(x)-g(x))\,dx = \int_0^3 f(x)\,dx - \int_0^3 g(x)\,dx = 7 - \Big[\int_0^2 g(x)\,dx + \int_2^3 g(x)\,dx\Big].$

$= 7 - (-3+5) = 5.$

21. $\int_0^3 (af(x)+bg(x))\,dx = 0 \Rightarrow a\int_0^3 f(x)\,dx + b\int_0^3 g(x)\,dx = 0.$

$\Rightarrow 7a + b(-3+5) = 0 \Rightarrow 7a + 2b = 0$, for a $= 2 \Rightarrow 14 + 2b = 0 \Rightarrow b = -7.$

23. $\int_5^0 (s(t)-r(t))\,dt = -\int_0^5 (s(t)-r(t))\,dt = \int_0^5 r(t)\,dt - \int_0^5 s(t)\,dt.$

$= \int_0^5 r(t)\,dt - \Big(\int_0^3 s(t)\,dt + \int_3^5 s(t)\,dt\Big) = 11 - (10+8) = -7.$

25. $\int_0^5 (ar(t)+bs(t))\,dt = 0 \Rightarrow a\int_0^5 r(t)\,dt + b\int_0^5 s(t)\,dt = 0 \Rightarrow 11a + 18b = 0.$

choose a $= 1 \Rightarrow 11 + 18b = 0 \Rightarrow b = \frac{-11}{18}.$

1.3 Riemann Sums

5. $\displaystyle\sum_{i=2}^{4} i^2 = \Big(\sum_{i=1}^{4} i^2\Big) - 1 = \frac{n(n+1)(2n+1)}{6} - 1 = \frac{4*5*9}{6} - 1 = 29.$

7. $\displaystyle\sum_{i=-2}^{2} \sin\left(\frac{i\pi}{2}\right) = \sin(-\pi) + \sin\left(\frac{-\pi}{2}\right) + \sin(0) + \sin\left(\frac{\pi}{2}\right) + \sin(\pi) = 0 - 1 + 0 + 1 + 0 = 0.$

9. $\displaystyle\sum_{i=1}^{6} -1^i i = (-1)^1 1 + (-1)^2 2 + (-1)^3 3 + (-1)^4 4 + (-1)^5 5 + (-1)^6 6.$

$= -1 + 2 - 3 + 4 - 5 + 6 = 3.$

11. $\displaystyle\sum_{i=0}^{5} (-1)^i \cos(i\pi) = (-1)^0\cos(0) + (-1)^1\cos(\pi) + (-1)^2\cos(2\pi) + (-1)^3\cos(3\pi) + .$

$(-1)^4\cos(4\pi) + (-1)^5\cos(5\pi) = 6.$

27. $\displaystyle\int_{-3}^{3} x^2\,dx.$

$$\Delta x = \frac{3-(-3)}{6} = 1.$$

$x_1 = -3 \qquad x_2 = -2.$

$x_3 = -1 \qquad x_4 = 0 \qquad x_5 = 1 \qquad x_6 = 2.$

$f(x_1) = 9 \qquad f(x_2) = 4 \qquad f(x_3) = 1.$

$f(x_4) = 0 \qquad f(x_5) = 1 \qquad f(x_6) = 4.$

$$\int_{-3}^{3} x^2\,dx = \sum_{i=1}^{6} f(x_i)\Delta x = 9 + 4 + 1 + 0 + 1 + 4 = 19.$$

29. $\displaystyle\int_{0}^{\pi} \sin x\,dx.$

$$\Delta x = \frac{\pi - 0}{6} = \frac{\pi}{6} \qquad x_i = \frac{i\pi}{6}.$$

$$x_1 = \frac{\pi}{6} \qquad x_2 = \frac{\pi}{3} \qquad x_3 = \frac{\pi}{2} \qquad x_4 = \frac{2\pi}{3} \qquad x_5 = \frac{5\pi}{6} \qquad x_6 = \pi.$$

$$f(x_1) = \sin(\frac{\pi}{6}) = \frac{1}{2} \qquad f(x_2) = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2} \qquad f(x_3) = \sin(\frac{\pi}{2}) = 1.$$

$$f(x_4) = \sin(\frac{2\pi}{3}) = \frac{\sqrt{3}}{2} \qquad f(x_5) = \sin(\frac{5\pi}{6}) = \frac{1}{2} \qquad f(x_6) = \sin(\pi) = 0.$$

$$\int_0^{\pi} \sin x\, dx = \Big(\frac{1}{2} + \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{3}}{2} + \frac{1}{2} + 0\Big)\frac{\pi}{6} = \frac{(2+\sqrt{3})\pi}{6}.$$

35. $\displaystyle\int_{-1}^{3} (3x-1)\, dx.$

$$\Delta x = \frac{3-(-1)}{n} = \frac{4}{n} \qquad x_i = -1 + \frac{4i}{n}. \qquad x_{i+1} = -1 + \frac{4(i+1)}{n}.$$

$$\Rightarrow x_i = \frac{x_i + x_{i+1}}{2} = -1 + \frac{4i+2}{n}.$$

$$\Rightarrow f(x_i) = f\Big(-1 + \frac{4i+2}{n}\Big) = -4 + \frac{12i+6}{n}.$$

$$\Rightarrow \int_{-1}^{3} (3x-1)\, dx = \sum_{i=1}^{n}\Big(-4 + \frac{12i+6}{n}\Big)\frac{4}{n} = \sum_{i=1}^{n} -\frac{16}{n} + \sum_{i=1}^{n}\frac{48i}{n^2} + \sum_{i=1}^{n}\frac{24}{n^2}.$$

$$= -16 + \frac{48}{n^2}\frac{n(n+1)}{2} + \frac{24}{n} = -16 + \frac{24n^2+24n}{n^2} + \frac{24}{n}.$$

$$n = 10 \Rightarrow \int_{-1}^{3} (3x-1)\, dx = -16 + \frac{2400+240}{100} + \frac{24}{10} = 12.8.$$

$$n = 100 \Rightarrow \int_{-1}^{3} (3x-1)\, dx = -16 + \frac{24 * 100^2 + 2400}{100^2} + \frac{24}{100} = 8.48.$$

$$n = 1000 \Rightarrow \int_{-1}^{3} (3x-1)\, dx = -16 + \frac{24 * 1000^2 + 24000}{1000^2} + \frac{24}{1000} = 8.048.$$

$$\lim_{n\to\infty}\Big(-16 + \frac{24n^2+24n}{n^2} + \frac{24}{n}\Big) = -16 + 24 = 8.$$

1.4 The Fundamental Theorem of Calculus

5. $\displaystyle\int_1^3 (3x^2 - 2x + 1)\, dx = \Big(\frac{3x^3}{3} - \frac{2x^2}{2} + x\Big)\Big|_1^3 = (x^3 - x^2 + x\Big|_1^3.$

$= (27 - 9 + 3) - (1 - 1 + 1) = 20.$

7. $\displaystyle\int_{-1}^{1} (x^3 - x^5)\, dx = \Big(\frac{x^4}{4} - \frac{x^6}{6}\Big)\Big|_{-1}^{1} = \Big(\frac{1}{4} - \frac{1}{6}\Big) - \Big(\frac{1}{4} - \frac{1}{6}\Big) = 0.$

9. $\displaystyle\int_0^{\frac{\pi}{4}} \sec^2 x\, dx = \tan x\Big|_0^{\frac{\pi}{4}} = \tan\frac{\pi}{4} - \tan 0 = 1.$

11. $\displaystyle\int_{-1}^{1} 5^x\,dx = \frac{5^x}{\ln 5}\Big|_{-1}^{1} = \frac{5-\frac{1}{5}}{\ln 5} = \frac{24}{5\ln 5}.$

13. $\displaystyle\int_0^{\pi} (2\cos x - 2\sin x)\,dx = (2\sin x + 2\cos x)\Big|_0^{\pi}.$

$= (2\sin\pi + 2\cos\pi) - (2\sin 0 + 2\cos 0) = -4.$

15. $\displaystyle\int_0^4 \sqrt{t}\,dt = \int_0^4 t^{\frac{1}{2}}\,dt = \frac{t^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^4 = \frac{2}{3}t^{\frac{3}{2}}\Big|_0^4 = \frac{16}{3}.$

17. $\displaystyle\int_1^8 x^{\frac{1}{3}}\,dx = \frac{3}{4}x^{\frac{4}{3}}\Big|_1^8 = \frac{3}{4}(16-1) = \frac{45}{4}.$

19. $\displaystyle\int_1^2 \frac{1}{x^2}\,dx = \int_1^2 x^{-2}\,dx = -\frac{1}{x}\Big|_1^2 = -\Big(\frac{1}{2}-1\Big) = \frac{1}{2}.$

21. $\displaystyle\int_0^1 x\,dx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}.$

23. $\displaystyle\int_0^1 x^3\,dx = \frac{1}{4}x^4\Big|_0^1 = \frac{1}{4}.$

25. $\displaystyle\int_{-4}^4 dx = x\Big|_{-4}^4 = 4 - (-4) = 8.$

27. $\displaystyle\int_{-2}^2 0\,dx = 0.$

31. $\displaystyle\int_{-2}^2 x^2\,dx = \frac{1}{3}x^3 = \frac{1}{3}(8+8) = \frac{16}{3}.$

$\displaystyle\int_{-2}^2 x^2\,dx = f(c)(2-(-2)) = 4f(c) \Rightarrow 4c^2 = \frac{16}{3} \Rightarrow c^2 = \frac{4}{3} \Rightarrow c = \frac{2}{\sqrt{3}}, \frac{-2}{\sqrt{3}}.$

33. $\displaystyle\int_0^{16} \sqrt{x}\,dx = \int_0^{16} x^{\frac{1}{2}}\,dx = \frac{2}{3}x^{\frac{3}{2}}\Big|_0^{1}6 = \frac{128}{3}.$

$\displaystyle f(c)(16-0) = \frac{128}{3} \Rightarrow 16\sqrt{c} = \frac{128}{3} \Rightarrow \sqrt{c} = \frac{8}{3} \Rightarrow c = \frac{64}{9}.$

35. $\displaystyle y = \sin x \Rightarrow Ave = \frac{1}{\pi - 0}\int_0^{\pi} \sin x\,dx = \frac{1}{\pi}(-\cos x)\Big|_0^{\pi} = \frac{-1}{\pi}(\cos\pi - \cos 0) = \frac{2}{\pi}.$

39. $\displaystyle g(t) = \frac{1}{t} \text{ on } [1,e] \Rightarrow Ave = \frac{1}{e-1}\int_1^e \frac{1}{t}\,dt = \frac{1}{e-1}\ln t\Big|_1^e = \frac{1}{e-1}(\ln e - \ln 1) = \frac{1}{e-1}.$

53. $\displaystyle F(x) = \int_2^{x^3+x} \frac{1}{t}\,dt \Rightarrow F'(x) = \frac{d}{dx}\int_2^{x^3+x} \frac{1}{t}\,dt = \frac{1}{x^3+x}\frac{d}{dx}(x^3+x) = \frac{3x^2+1}{x^3+x}.$

55. $\displaystyle F(x) = \int_x^{x^2} (t+2)\,dt = \int_x^0 (t+2)\,dt + \int_0^{x^2} (t+2)\,dt.$

$\displaystyle = -\int_0^x (t+2)\,dt + \int_0^{x^2} (t+2)\,dt \Rightarrow F'(x) = \frac{d}{dx}\Big[\int_0^{x^2} (t+2)\,dt - \int_0^x (t+2)\,dt\Big].$

$\displaystyle = (x^2+2)\frac{d}{dx}(x^2) - (x+2)\frac{d}{dx}(x) = (x^2+2)2x - (x+2) = 2x^3 + 3x - 2.$

Chapter 2

Techniques of Integration

2.1 Substitution Method

3. $x^3 - 5 = u \Rightarrow 3x^2dx = du \Rightarrow \int u^7\,du = \frac{1}{8}u^8 + C = \frac{1}{8}(x^3-5)^8 + C.$

5. $x^2 + 1 = u \Rightarrow 2xdx = du \Rightarrow xdx = \frac{du}{2}.$

   $\Rightarrow \int \frac{u^8}{2}\,du = \frac{1}{18}u^9 + C = \frac{1}{18}(x^2+1)^9 + C.$

11. $\sqrt{x} = u \Rightarrow \frac{1}{2\sqrt{x}}dx = du \Rightarrow \frac{1}{\sqrt{x}}dx = 2du.$

    $\Rightarrow \int 2e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C.$

13. $\frac{1}{x} + 1 = u \Rightarrow \frac{-1}{x^2}dx = du.$

    $\Rightarrow \int -u\,du = \frac{-1}{2}u^2 + C = \frac{-1}{2}\left(\frac{1}{x}+1\right)^2 + C.$

15. $\sin x = u \Rightarrow \cos x dx = du.$

    $\Rightarrow \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3 x + C.$

17. $4 - x = u \Rightarrow -dx = du.$

    $\Rightarrow \int -\sec 2u\,du = -\tan u + C = -\tan(4-x) + C.$

19. $\tan x = u \Rightarrow \sec^2 x dx = du.$

    $\Rightarrow \int u^2\,du = \frac{1}{3}u^3 + C = -\frac{1}{3}\tan^3 x + C.$

25. $x^3 = u \Rightarrow 3x^2dx = du \Rightarrow x^2dx = \frac{1}{3}du.$

$$\Rightarrow \int \frac{1}{3}e^u\,du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C.$$

31. $\ln x = u \Rightarrow \frac{1}{x}dx = du.$

$$\Rightarrow \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\ln^2 x + C.$$

35. $\displaystyle\int \frac{x^2+3x+1}{x}\,dx = \int \frac{x^2}{x} + \frac{3x}{x} + \frac{1}{x}\,dx = \int x + 3 + \frac{1}{x}\,dx = \frac{1}{2}x^2 + 3x + \ln|x| + C.$

37. $\displaystyle\int \frac{x^3-1}{x+1}\,dx = \int \left((x^2-x+1) - \frac{2}{x+1}\right)dx = \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - \int \frac{2}{x+1}\,dx.$

$$= \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - 2\ln|x+1| + C.$$

51. $\displaystyle\int \frac{x^2}{(x^3+3)^2}\,dx.$

$$x^3 + 3 = u \Rightarrow 3x^2dx = du \Rightarrow x^2dx = \frac{du}{3}.$$

$$\Rightarrow \int \frac{1}{3u^2}\,du = \frac{-1}{3u} + C = \frac{-1}{3(x^3+3} + C.$$

53. $1 - x^2 = u \Rightarrow -2xdx = du \Rightarrow xdx = \frac{-du}{2}.$

$$\Rightarrow \int \frac{-1}{2\sqrt{u}}\,du = -\sqrt{u} + C = -\sqrt{1-x^2} + C.$$

79. $1 - x^2 = u \Rightarrow -2xdx = du.$

$x = 0 \Rightarrow u = 1.$

$x = 1 \Rightarrow u = 0.$

$$\Rightarrow -\int_1^0 u^4\,du = \frac{-1}{5}u^5\Big|_1^0 = \frac{1}{5}.$$

2.2 Integration by Parts

5. $\displaystyle\int xe^{-x}\,dx.$

$u = x \Rightarrow du = dx. \qquad e^{-x}dx = dv \Rightarrow -e^{-x} = v.$

$$\Rightarrow \int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} = e^{-x}(1+x).$$

7. $\displaystyle\int x^3 \sin x \, dx.$

$u = x^3 \Rightarrow du = 3x^2 dx. \qquad \sin x dx = dv \Rightarrow -\cos x = v.$

$$\Rightarrow \int x^3 \sin x \, dx = -x^3 \cos x + 3 \int x^2 \cos x \, dx.$$

$u = x^2 \Rightarrow du = 2x dx. \qquad \cos x dx = dv \Rightarrow \sin x = v.$

$$\Rightarrow \int x^3 \sin x \, dx = -x^3 \cos x + 3\Big(x^2 \sin x - 2 \int x \sin x \, dx\Big).$$

$u = x \Rightarrow du = dx. \qquad \sin x dx = dv \Rightarrow -\cos x = v.$

$$\Rightarrow \int x^3 \sin x \, dx = -x^3 \cos x + 3\Big(x^2 \sin x - 2\Big(\sin x - x \cos x\Big)\Big).$$

9. $\displaystyle\int x^3 e^x \, dx.$

$u = x^3 \Rightarrow du = 3x^2 dx. \qquad e^x dx = dv \Rightarrow e^x = v.$

$$\Rightarrow \int x^3 e^x \, dx = x^3 e^x - 3 \int x^2 e^x \, dx.$$

$u = x^2 \Rightarrow du = 2x dx. \qquad e^x dx = dv \Rightarrow e^x = v.$

$$\Rightarrow \int x^3 e^x \, dx = x^3 e^x - 3\Big(x^2 e^x - 2 \int x e^x \, dx\Big).$$

$u = x \Rightarrow du = dx. \qquad e^x dx = dv \Rightarrow e^x = v.$

$$\Rightarrow \int x^3 e^x \, dx = x^3 e^x - 3\Big(x^2 e^x - 2\Big(x e^x - e^x\Big)\Big).$$

11. $\displaystyle\int e^x \sin x \, dx.$

$u = \sin x \Rightarrow du = \cos x dx. \qquad e^x dx = dv \Rightarrow e^x = v.$

$$\Rightarrow \int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx.$$

$u = \cos x \Rightarrow du = -\sin x dx. \qquad e^x dx = dv \Rightarrow e^x = v.$

$$\Rightarrow \int e^x \sin x \, dx = e^x \sin x - \Big(e^x \cos x + \int e^x \sin x \, dx\Big).$$

$$\Rightarrow 2\int e^x \sin x\,dx = e^x \sin x - e^x \cos x \Rightarrow \int e^x \sin x\,dx = \frac{1}{2}e^x\Big(\sin x - \cos x\Big).$$

13. $\displaystyle\int e^{2x} \sin 3x\,dx.$

$$u = \sin 3x \Rightarrow du = 3\cos 3x dx. \qquad e^{2x}dx = dv \Rightarrow \frac{1}{2}e^x = v.$$

$$\Rightarrow \int e^{2x}\sin 3x\,dx = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int e^{2x}\cos 3x\,dx.$$

$$u = \cos 3x \Rightarrow du = -3\sin 3x dx. \qquad e^{2x}dx = dv \Rightarrow \frac{1}{2}e^x = v.$$

$$\Rightarrow \int e^{2x}\sin 3x\,dx = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\Big(\frac{1}{2}e^x\cos 3x + \frac{3}{2}\int e^{2x}\sin 3x\,dx\Big).$$

$$\Rightarrow \int e^{2x}\sin 3x\,dx = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^x\cos 3x - \frac{9}{4}\int e^{2x}\sin 3x\,dx\Big).$$

$$\Rightarrow \frac{13}{4}\int e^{2x}\sin 3x\,dx = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^x\cos 3x$$

$$\Rightarrow \int e^{2x}\sin 3x\,dx = \frac{4}{26}e^{2x}\sin 3x - \frac{3}{13}e^x\cos 3x.$$

21. $\displaystyle\int (x-2)\ln x\,dx.$

$$u = \ln x \Rightarrow du = \frac{1}{x}dx. \qquad (x-2)dx = dv \Rightarrow \frac{1}{2}(x-2)^2 = v.$$

$$\Rightarrow \int (x-2)\ln x\,dx = \frac{1}{2}(x-2)^2\ln x - \int \frac{1}{2}(x-2)^2\frac{1}{x}\,dx.$$

$$= \frac{1}{2}(x-2)^2\ln x - \frac{1}{2}\int \frac{x^2-4x+4}{x}\,dx.$$

$$= \frac{1}{2}(x-2)^2\ln x - \frac{1}{2}\int \Big(x - 4 + \frac{4}{x}\Big)\,dx = \frac{1}{2}(x-2)^2\ln x - \frac{1}{4}x^2 + 2x - 2\ln x + C.$$

23. $\displaystyle\int x\ln x^2\,dx.$

$$u = x^2 \Rightarrow du = 2xdx \Rightarrow \int x\ln x^2\,dx = \int \frac{1}{2}\ln u\,du.$$

$$w = \ln u \Rightarrow dw = \frac{1}{u}du. \qquad du = dv \Rightarrow u = v.$$

$$= \frac{1}{2}\Big(u\ln u - \int \frac{u}{u}\,du\Big) = \frac{1}{2}u\Big(\ln u - 1\Big) = \frac{1}{2}x^2\Big(\ln x^2 - 1\Big) + C.$$

39. $\displaystyle\int_0^{\pi} x\sin x\,dx.$

$u = x \Rightarrow du = dx. \qquad \sin x dx = dv \Rightarrow -\cos x = v.$

$$\Rightarrow \int_0^{\pi} x\sin x\,dx = -x\cos x\Big|_0^{\pi} + \int_0^{\pi}\cos x\,dx = x\cos x\Big|_0^{\pi} + \sin x\Big|_0^{\pi}.$$

$$= \Big(-\pi\cos\pi + 0\cos 0\Big) + \Big(\sin\pi - \sin 0\Big) = \pi.$$

41. $\displaystyle\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} x^2\sin x\,dx.$

$u = x^2 \Rightarrow du = 2xdx. \qquad \sin x dx = dv \Rightarrow -\cos x = v.$

$$\Rightarrow \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} x^2\sin x\,dx = -x^2\cos x + 2\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} x\cos x\,dx.$$

$u = x \Rightarrow du = dx. \qquad \cos x dx = dv \Rightarrow -\sin x = v.$

$$\Rightarrow \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} x^2\sin x\,dx = \Big(-x^2\cos x + 2\Big(x\sin x - \cos x\Big)\Big)\Big|_{\frac{-\pi}{4}}^{\frac{\pi}{4}} = 0.$$

43. $\displaystyle\int_0^{\ln\sqrt{2}} xe^{x^2}\,dx.$

$u = x^2 \Rightarrow du = 2xdx.$

$x = 0 \Rightarrow u = 0 \qquad x = \sqrt{\ln 2} \Rightarrow u = \ln 2.$

$$\Rightarrow \int_0^{\ln\sqrt{2}} xe^{x^2}\,dx = \frac{1}{2}\int_0^{\ln 2} e^u\,du = \frac{1}{2}e^u\Big|_0^{\ln 2} = \frac{1}{2}.$$

45. $\displaystyle\int_1^2 xe^{-2x}\,dx.$

$u = x \Rightarrow du = dx. \qquad e^{-2x}dx = dv \Rightarrow -\frac{1}{2}e^{-2x} = v.$

$$\Rightarrow \int_1^2 xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x}\Big|_1^2 + \frac{1}{2}\int_1^2 e^{-2x}\,dx = \frac{-1}{2}xe^{-2x}\Big|_1^2 - \frac{-1}{4}e^{-2x}\Big|_1^2.$$

$$= -\frac{1}{2}\Big(2e^{-4} - e^{-2}\Big) - \frac{1}{4}\Big(e^{-4} - e^{-2}\Big) = -\frac{5}{4}e^{-4} + \frac{3}{4}e^{-2}.$$

47. $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2x}\cos x\,dx.$

$u = \cos x \Rightarrow -\sin x dx = du. \qquad e^{2x}dx = dv \Rightarrow -\frac{1}{2}e^{2x} = v.$

$$\Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2x}\cos x\,dx = \frac{1}{2}e^{2x}\cos x + \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2x}\sin x\,dx.$$

$u = \sin x \Rightarrow -\cos x dx = du. \qquad e^{2x}dx = dv \Rightarrow \frac{1}{2}e^{2x} = v.$

$$\Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2x}\cos x\,dx = \frac{1}{2}e^{2x}\cos x + \frac{1}{2}\Big(\frac{1}{2}e^{2x}\sin x - \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2x}\cos x\,dx\Big).$$

$$\Rightarrow \frac{5}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2x}\cos x\,dx = \frac{1}{2}e^{2x}\cos x + \frac{1}{4}e^{2x}\sin x.$$

$$\Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2x}\cos x\,dx = \frac{4}{10}e^{2x}\cos x + \frac{4}{20}e^{2x}\sin x\Big|_{\frac{-\pi}{2}}^{\frac{\pi}{2}}.$$

$$\Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2x}\cos x\,dx = \frac{4}{20}\Big(e^{\pi} + e^{-\pi}\Big).$$

2.3 Trigonometric Integrals

5. $\displaystyle\int \sin^3 x\cos x\,dx.$

   Let $\sin x = u \Rightarrow \cos x dx = du \Rightarrow \displaystyle\int \sin^3 x\cos x\,dx.$

   $= \displaystyle\int u^3\,du = \frac{1}{4}u^4 + C = \frac{1}{4}\sin^4 x + C.$

7. $\displaystyle\int \sin^3 x\cos^3 x\,dx = \int \sin^3 x\cos^2 x\cos x\,dx = \int \sin^3 x(1-\sin^2 x)\cos x\,dx.$

   Let $\sin x = u \Rightarrow \cos x dx = du \Rightarrow \displaystyle\int \sin^3 x(1-\sin^2 x)\cos x\,dx = \int u^3(1-u^2)\,du.$

   $= \displaystyle\int (u^3 - u^5)\,du = \frac{1}{4}u^4 - \frac{1}{6}u^6 + C = \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C.$

9. $\displaystyle\int \sin^2 x\cos^7 x\,dx = \int \sin^2 x\cos^6 x\cos x\,dx = \int \sin^2 x(1-\sin^2 x)^3\cos x\,dx.$

   Let $\sin x = u \Rightarrow \cos x dx = du \Rightarrow \displaystyle\int \sin^2 x(1-\sin^2 x)^3\cos x\,dx.$

   $= \displaystyle\int u^2(1-u^2)^3\,du = \int u^2(1-3u^2+3u^4-u^6)\,du = \int u^2 - 3u^4 + 3u^6 - u^8\,du.$

$$= \frac{1}{3}u^3 - \frac{3}{5}u^5 + \frac{3}{7}u^7 - \frac{1}{9}u^9 + C = \frac{1}{3}\sin^3 x - \frac{3}{5}\sin^5 x + \frac{3}{7}\sin^7 x - \frac{1}{9}\sin^9 x + C.$$

11. $\displaystyle\int \sin 5x \cos 3x \, dx = \frac{1}{2}\int (\sin 2x + \sin 8x)\, dx = \frac{1}{2}\Big(-\frac{1}{2}\cos 2x - \frac{1}{8}\cos 8x\Big) + C.$

$$= -\frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + C.$$

17. $\displaystyle\int \tan^4 x \sec^2 x \, dx.$

$$n = 2 \Rightarrow k = 1 \Rightarrow \int \tan^4 x \sec^2 x \, dx = \int u^4(1+u^2)^{1-1}\, du = \int u^4 \, du = \frac{1}{5}u^5 + C.$$

$$= \frac{1}{5}\tan^5 x + C.$$

27. $\displaystyle\int_0^{\pi} \sin x \cos^4 x \, dx.$

$$m = 1 \Rightarrow k = 0 \Rightarrow \int \sin x \cos^4 x \, dx = -\int (1-u^2)^0 u^4 \, du = -\int u^4 \, du = -\frac{1}{5}u^5.$$

$$\Rightarrow \int_0^{\pi} \sin x \cos^4 x \, dx = -\frac{1}{5}\cos^5 x\Big|_0^{\pi} = \frac{2}{5}.$$

29. $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cos^7 x \, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cos^6 x \cos x \, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x (1-\sin^2 x)^3 \cos x \, dx.$

Let $u = \sin x \Rightarrow du = \cos x dx.$

$$\Rightarrow \int_{-1}^{1} u^2(1-u^2)^3 \, du = \Big(\frac{1}{3}u^3 - \frac{3}{5}u^5 + \frac{3}{7}u^7 - \frac{1}{9}u^9\Big)\Big|_{-1}^{1} = \frac{32}{315}.$$

31. $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \cos 2x \, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x (1 - 2\sin^2 x)\, dx.$

Let $u = \sin x \Rightarrow du = \cos x dx.$

$$\Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x(1-2\sin^2 x)\, dx = \int_{-1}^{1} (1-2u^2)\, du = \Big(u - \frac{2}{3}u^3\Big)\Big|_{-1}^{1} = \frac{2}{3}.$$

2.4 Trigonometric Substitutions

5. $\displaystyle\int \sqrt{x^2+1}\, dx = \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln|x + \sqrt{x^2+1}| + C.$

7. $\displaystyle\int \sqrt{1-x^2}\, dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arctan\frac{x}{\sqrt{1-x^2}} + C.$

9. $\displaystyle\int \sqrt{x^2-1}\, dx = \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln|x + \sqrt{x^2-1}| + C.$

11. $\displaystyle\int \sqrt{4x^2+1}\, dx = \int \sqrt{(2x)^2+1}\, dx = \frac{2x}{2}\sqrt{4x^2+1} + \frac{1}{2}\ln|2x + \sqrt{4x^2+1}| + C.$

13. $\displaystyle\int \sqrt{(4x)^2-1}\, dx = \frac{1}{2}4x\sqrt{(4x)^2-1} - \frac{1}{2}\ln|4x + \sqrt{(4x)^2-1}| + C.$

15. $\displaystyle\int \frac{3}{\sqrt{7-x^2}}\, dx = 3\int \frac{1}{\sqrt{(\sqrt{7})^2 - x^2}}\, dx = \arcsin\frac{x}{\sqrt{7}} + C.$

16. $\displaystyle\int \frac{5}{\sqrt{x^2-8}}\,dx = 5\int \frac{1}{\sqrt{x^2-(\sqrt{8})^2}}\,dx = 5\ln|x+\sqrt{x^2-8}|+C.$

25. $\displaystyle\int \frac{\sqrt{5-x^2}}{7x^2}\,dx = \frac{1}{7}\int \frac{\sqrt{(\sqrt{5})^2-x^2}}{x^2}\,dx = \frac{1}{7}\Big(-\frac{1}{x}\sqrt{5-x^2}-\arcsin\frac{x}{\sqrt{5}}\Big)+C.$

27. $\displaystyle\int_{-1}^{1}\sqrt{1-x^2}\,dx = \int_{-1}^{1}\sqrt{1^2-x^2}\,dx = \frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\arcsin(x)\Big|_{-1}^{1}.$

$\displaystyle = \frac{1}{2}\sqrt{1-1}+\frac{1}{2}\arcsin(1)-\Big(\frac{1}{2}\sqrt{1--1^2}+\frac{1}{2}\arcsin(-1)\Big)=\frac{\pi}{2}.$

29. $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2(x)\cos^7(x)\,dx = -\frac{\sin(x)\cos^8(x)}{2+7}+\frac{2-1}{2+7}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^7(x)\,dx.$

$\displaystyle = \frac{\sin(x)\cos^8(x)}{9}+\frac{1}{9}\Big(\frac{1}{7}\cos^6(x)\sin(x)+\frac{6}{7}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^5(x)\,dx\Big).$

$\displaystyle = \frac{\sin(x)\cos^8(x)}{9}+\frac{1}{9}\Big(\frac{1}{7}\cos^6(x)\sin(x)+\frac{6}{7}\Big(\frac{1}{5}\cos^4(x)\sin(x)+\frac{4}{5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^3(x)\,dx\Big)\Big).$

$\displaystyle = \frac{\sin(x)\cos^8(x)}{9}+\frac{1}{63}\cos^6(x)\sin(x)+\frac{6}{63}\Big(\frac{1}{5}\cos^4(x)\sin(x)+\frac{4}{5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^3(x)\,dx\Big).$

$\displaystyle = \frac{\sin(x)\cos^8(x)}{9}+\frac{1}{63}\cos^6(x)\sin(x)+\frac{6}{315}\cos^4(x)\sin(x)+\frac{24}{315}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^3(x)\,dx.$

$\displaystyle = \frac{\sin(x)\cos^8(x)}{9}+\frac{1}{63}\cos^6(x)\sin(x)+\frac{6}{315}\cos^4(x)\sin(x)+\frac{24}{315}\Big(\frac{1}{3}\cos^2(x)\sin(x)+$
$\displaystyle \frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(x)\,dx\Big).$

$\displaystyle = \Big(\frac{\sin(x)\cos^8(x)}{9}+\frac{1}{63}\cos^6(x)\sin(x)+\frac{6}{315}\cos^4(x)\sin(x)+\frac{24}{315}\Big(\frac{1}{3}\cos^2(x)\sin(x)+$

$\displaystyle \frac{2}{3}\sin(x)\Big)\Big)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{48}{945}\Big(\sin(\frac{\pi}{2})-\sin(\frac{-\pi}{2})\Big)=\frac{96}{945}=\frac{32}{315}.$

31. $\displaystyle\int_{-1}^{1}\sqrt{3^2-x^2}\,dx = \Big(\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\arcsin(\frac{x}{3})\Big)\Big|_{-1}^{1} = 9\arcsin\left(\frac{1}{3}\right)+\sqrt{8}.$

2.5 Partial Fractions Decomposition

7. $\displaystyle\int \frac{7x+7}{x^2+3x-10}\,dx = \int \frac{7x+7}{(x+5)(x-2)}\,dx.$

$$= \int \frac{A}{(x+5)} + \frac{B}{(x-2)}\,dx = \int \frac{A(x-2)+B(x+5)}{(x+5)(x-2)}\,dx.$$

$$\Rightarrow A(x-2)+B(x+5) = 7x+7 \Rightarrow x(A+B)+(5B-2A) = 7x+7.$$

$$\begin{cases} A+B=7 \\ 5B-2A=7 \end{cases} \Rightarrow \begin{cases} A=4 \\ B=3 \end{cases}.$$

$$\Rightarrow \int \frac{7x+7}{x^2+3x-10}\,dx = \int \frac{4}{(x+5)} + \frac{3}{(x-2)}\,dx = 4\ln|x+5| + 3\ln|x-2| + C.$$

9. $$\int \frac{-4}{3x^2-12}\,dx = \frac{-4}{3}\int \frac{1}{(x-2)(x+2)}\,dx.$$

$$= \frac{-4}{3}\int \frac{A}{(x-2)} + \frac{B}{(x+2)}\,dx = \frac{-4}{3}\int \frac{A(x+2)+B(x-2)}{(x+2)(x-2)}\,dx.$$

$$\Rightarrow A(x+2)+B(x-2) = 1 \Rightarrow x(A+B)+(2A-2B) = 1.$$

$$\begin{cases} A+B=0 \\ 2A-2B=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{4} \\ B=-\frac{1}{4} \end{cases}.$$

$$\Rightarrow \int \frac{-4}{3x^2-12}\,dx = -\frac{4}{3}\int \frac{1}{4(x-2)} - \frac{1}{4(x+2)}\,dx.$$

$$= -\frac{1}{3}\ln|x-2| + \frac{1}{3}\ln|x+2| + C.$$

13. $$\int \frac{-12x^2-x+33}{(x-1)(x+3)(3-2x)}\,dx = \int \Big(\frac{A}{(x-1)} + \frac{B}{x+3} + \frac{C}{3-2x}\Big)\,dx.$$

$$\Rightarrow A(x+3)(3-2x) + B(x-1)(3-2x) + C(x-1)(x+3) = -12x^2 - x + 33.$$

$$\Rightarrow x^2(-2A-2B+C) + x(-3A+5B+2C) + 9A-3B-3C = -12x^2 - x + 33.$$

$$\begin{cases} -2A-2B+C=-12 \\ -3A+5B+2C=-1 \\ 9A-3B-3C=33 \end{cases} \Rightarrow \begin{cases} A=5 \\ B=2 \\ C=2 \end{cases}.$$

$$\int \Big(\frac{A}{(x-1)} + \frac{B}{x+3} + \frac{C}{3-2x}\Big)\,dx = \int \Big(\frac{5}{(x-1)} + \frac{2}{x+3} + \frac{2}{3-2x}\Big)\,dx.$$

$$= 5\ln|x-1| + 2\ln|x+3| - \ln|3-2x| + C.$$

15. $$\int \frac{x^2+x+1}{x^2+x-2}\,dx = \int \frac{x^2+x-2+3}{x^2+x-2}\,dx = \int \Big(1 + \frac{3}{x^2+x-2}\Big)\,dx.$$

$$= \int 1\, dx + \int \frac{3}{x^2+x-2}\, dx = x + \int \frac{3}{(x-1)(x+2)}\, dx.$$

$$= x + \int \Big(\frac{A}{(x-1)} + \frac{B}{(x+2)}\Big)\, dx$$

$$\Rightarrow A(x+2) + B(x-1) = 3 \Rightarrow x(A+B) + (2A-B) = 3.$$

$$\begin{cases} A+B=0 \\ 2A-B=3 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-1 \end{cases}.$$

$$\Rightarrow x + \int \Big(\frac{A}{(x-1)} + \frac{B}{(x+2)}\Big)\, dx = x + \int \Big(\frac{1}{(x-1)} + \frac{-1}{(x+2)}\Big)\, dx.$$

$$= x + \ln|x-1| - \ln|x+2| + C.$$

17. $\displaystyle\int \frac{2x^2-4x+6}{x^2-2x+3}\, dx = \int \frac{2(x^2-2x+3)}{x^2-2x+3}\, dx = \int 2\, dx = 2x + C.$

21. $\displaystyle\int \frac{6x^2+8x-4}{(x-3)(x^2+6x+10)}\, dx = \int = \int \Big(\frac{A}{x-3} + \frac{Bx+C}{x^2+6x+10}\Big)\, dx.$

$$\Rightarrow A(x^2+6x+10) + (Bx+C)(x-3) = 3.$$

$$\Rightarrow x^2(A+B) + x(6A-3B+C) + (10A-3C) = 6x^2+8x-4.$$

$$\begin{cases} A+B=6 \\ 6A-3B+C=8 \\ 10A-3C=-4 \end{cases} \Rightarrow \begin{cases} A=2 \\ B=4 \\ C=8 \end{cases}.$$

$$\Rightarrow \int \Big(\frac{A}{x-3} + \frac{Bx+C}{x^2+6x+10}\Big)\, dx = \int \Big(\frac{2}{x-3} + \frac{4x+8}{x^2+6x+10}\Big)\, dx =.$$

$$\int \frac{2}{x-3}\, dx + \int \frac{4x+8}{x^2+6x+10}\, dx = \int \frac{2}{x-3}\, dx + \int \frac{4x+12-4}{x^2+6x+10}\, dx.$$

$$= \int \frac{2}{x-3}\, dx + 2\int \frac{2x+6}{x^2+6x+10}\, dx - \int \frac{4}{x^2+6x+10}\, dx.$$

$$= 2\ln|x-3| + 2\ln|x^2+6x+10| - 4\int \frac{1}{(x+3)^2+1}\, dx.$$

$$= 2\ln|x-3| + 2\ln|x^2+6x+10| - 4\arctan(x+3) + C.$$

23. $\displaystyle\int \frac{x^2-20x-69}{(x-7)(x^2+2x+17)}\, dx = \int = \int \Big(\frac{A}{x-7} + \frac{Bx+C}{x^2+2x+17}\Big)\, dx.$

$$\Rightarrow A(x^2+2x+17) + (Bx+C)(x-7) = 3.$$

$$\Rightarrow x^2(A+B) + x(2A-7B+C) + (17A-7C) = x^2 - 20x - 69.$$

$$\begin{cases} A+B=1 \\ 2A-7B+C=-20 \\ 17A-7C=-69 \end{cases} \Rightarrow \begin{cases} A=-2 \\ B=3 \\ C=5 \end{cases}.$$

$$\Rightarrow \int \Big(\frac{A}{x-7} + \frac{Bx+C}{x^2+2x+17}\Big)\, dx = \int \Big(\frac{-2}{x-7} + \frac{3x+5}{x^2+2x+17}\Big)\, dx.$$

$$= -2\ln|x-7| + \int \frac{3x+3+2}{x^2+2x+17}\, dx.$$

$$= -2\ln|x-7| + 3\int \frac{(x+1)}{x^2+2x+1+16}\, dx + \int \frac{2}{x^2+2x+1+16}\, dx.$$

$$= -2\ln|x-7| + 3\int \frac{(x+1)}{(x+1)^2+4^2}\, dx + 2\int \frac{1}{(x+1)^2+4^2}\, dx.$$

$$= -2\ln|x-7| + \frac{3}{2}\ln|x^2+2x+17| + \arctan(\frac{x+1}{4}) + C.$$

25. $$\int \frac{6x^2+45x+121}{(x+2)(x^2+10x+27)}\, dx = \int = \int \Big(\frac{A}{x+2} + \frac{Bx+C}{x^2+10x+27}\Big)\, dx.$$

$$\Rightarrow A(x^2+10x+27) + (Bx+C)(x+2) = 6x^2+45x+121.$$

$$\Rightarrow x^2(A+B) + x(10A+2B+C) + (27A+2C) = 6x^2+45x+121.$$

$$\begin{cases} A+B=6 \\ 10A+2B+C=45 \\ 27A+2C=121 \end{cases} \Rightarrow \begin{cases} A=5 \\ B=1 \\ C=-7 \end{cases}.$$

$$\Rightarrow \int \Big(\frac{A}{x+2} + \frac{Bx+C}{x^2+10x+27}\Big)\, dx = \int \Big(\frac{5}{x+2} + \frac{x-7}{x^2+10x+27}\Big)\, dx.$$

$$\Rightarrow \int \Big(\frac{A}{x+2} + \frac{Bx+C}{x^2+10x+27}\Big)\, dx = \int \Big(\frac{5}{x+2} + \frac{x-7}{x^2+10x+27}\Big)\, dx.$$

$$= 5\ln|x+2| + \int \frac{x+5-12}{x^2+10x+25+2}\, dx.$$

$$= 5\ln|x+2| + \int \frac{x+5}{(x+5)^2+2}\, dx - 12\int \frac{1}{(x+5)^2+2}\, dx.$$

$$= 5\ln|x+2| + \frac{1}{2}\ln|x^2+10x+27| - 6\sqrt{2}\arctan\Big(\frac{x+5}{\sqrt{2}}\Big) + C.$$

27. $\displaystyle\int \frac{14x+6}{(3x+2)(x+4)}\,dx = \int = \int \Big(\frac{A}{3x+2} + \frac{B}{x+4}\Big)\,dx.$

$\Rightarrow A(x+4) + B(3x+2) = 14x+6.$

$\Rightarrow x(A+3B) + (4A+2B) = 14x+6.$

$$\begin{cases} A+3B = 14 \\ 4A+2B = 6 \end{cases} \Rightarrow \begin{cases} A = -1 \\ B = 5 \end{cases}.$$

$$\int_0^5 \Big(\frac{A}{3x+2} + \frac{B}{x+4}\Big)\,dx = \int_0^5 \Big(\frac{-1}{3x+2} + \frac{5}{x+4}\Big)\,dx.$$

$$= -\frac{1}{3}\ln|3x+2| + 5\ln|x+4|\Big|_0^5 = 5\ln\Big(\frac{9}{4}\Big) - \frac{1}{3}\ln\Big(\frac{17}{2}\Big).$$

29. $\displaystyle\int_0^1 \frac{x}{(x+1)(x^2+2x+1)}\,dx = \int_0^1 \frac{x+1-1}{(x+1)(x^2+2x+1)}\,dx.$

$$= \int_0^1 \frac{x+1}{(x+1)(x^2+2x+1)}\,dx - \int_0^1 \frac{1}{(x+1)(x^2+2x+1)}\,dx.$$

$$= \int_0^1 \frac{1}{(x+1)^2}\,dx - \int_0^1 \frac{1}{(x+1)^3}\,dx = \Big(-\frac{1}{x+1} + \frac{1}{2(x+1)^2}\Big)\Big|_0^1 = \frac{1}{8}.$$

2.6 Improper Integration

7. $\displaystyle\int_0^\infty e^{5-2x}\,dx = \lim_{t\to\infty}\int_0^t e^{5-2x}\,dx = \lim_{t\to\infty}\Big(-\frac{1}{2}e^{5-2x}\Big)\Big|_0^t.$

$$= -\frac{1}{2}\lim_{t\to\infty}(e^{5-2t} - e^5) = \frac{e^5}{2}.$$

11. $\displaystyle\int_{-\infty}^0 2^x\,dx = \lim_{t\to-\infty}\int_t^0 2^x\,dx = \frac{1}{\ln 2}\lim_{t\to-\infty}(2^0 - 2^t) = \frac{1}{\ln 2}.$

13. $\displaystyle\int_{-\infty}^\infty \frac{x}{x^2+1}\,dx = \int_{-\infty}^0 \frac{x}{x^2+1}\,dx + \int_0^\infty \frac{x}{x^2+1}\,dx = \lim_{t\to-\infty}\int_t^0 \frac{x}{x^2+1}\,dx + \lim_{t\to\infty}\int_0^t \frac{x}{x^2+1}\,dx.$

$$= \lim_{t\to-\infty}\frac{1}{2}\ln|x^2+1|\Big|_t^0 + \lim_{t\to\infty}\frac{1}{2}\ln|x^2+1|\Big|_0^t = \infty$$

15. $\displaystyle\int_2^\infty \frac{1}{(x-1)^2}\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{(x-1)^2}\,dx = \lim_{t\to\infty}\frac{1}{x-1}\Big|_2^t = \lim_{t\to\infty}\Big(\frac{1}{t-1} - 1\Big) = -1.$

19. $\displaystyle\int_{-1}^1 \frac{1}{x}\,dx = \int_{-1}^0 \frac{1}{x}\,dx + \int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0}\int_{-1}^t \frac{1}{x}\,dx + \lim_{t\to 0}\int_t^1 \frac{1}{x}\,dx.$

$$= \lim_{t\to 0} \ln|x|\Big|_{-1}^{t} + \lim_{t\to 0} \ln|x|\Big|_{t}^{1} = \infty.$$

23. $$\int_0^\infty xe^{-x}\,dx = \lim_{t\to\infty}\int_0^t xe^{-x}\,dx = \lim_{t\to\infty}\Big(xe^{-x}\Big|_0^t + \int_0^t e^{-x}\,dx\Big).$$

$$= \lim_{t\to\infty}\Big(xe^{-x} - e^{-x}\Big)\Big|_0^t = \lim_{t\to\infty}\Big(te^{-t} - e^{-t} + e^0\Big) = 1.$$

25. $$\int_{-\infty}^\infty xe^{-x^2}\,dx = \int_{-\infty}^0 xe^{-x^2}\,dx + \int_0^\infty xe^{-x^2}\,dx = \lim_{t\to-\infty} -\frac{1}{2}e^{-x^2}\Big|_t^0 + \lim_{t\to\infty} -\frac{1}{2}e^{-x^2}\Big|_0^t.$$

$$= -\frac{1}{2}(1-0)) - \frac{1}{2}(0-1) = 0.$$

27. $$\int_0^1 x\ln x\,dx = \lim_{t\to 0}\int_t^1 x\ln x\,dx = \lim_{t\to 0}\Big(\frac{1}{2}x^2\ln x - \frac{1}{2}\int_t^1 x\,dx\Big).$$

$$= \lim_{t\to 0}\Big(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\Big)\Big|_t^1 = -\frac{1}{4}.$$

33. $$\int_0^\infty e^{-x}\cos x\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\cos x\,dx.$$

$$\int_0^t e^{-x}\cos x\,dx = -e^{-x}\cos x - \int_0^t e^{-x}\sin x\,dx.$$

$$= -e^{-x}\cos x - \Big(-e^{-x}\sin x + \int_0^t e^{-x}\cos x\,dx\Big).$$

$$\Rightarrow \int_0^t e^{-x}\cos x\,dx = \frac{1}{2}e^{-x}(\sin x - \cos x)\Big|_0^t.$$

$$\Rightarrow \int_0^\infty e^{-x}\cos x\,dx = \frac{1}{2}\lim_{t\to\infty}\Big(e^{-x}(\sin x - \cos x)\Big)\Big|_0^t = \frac{1}{2}.$$

2.7 Numerical Integration

5. **(a)**:

$$\int_0^{10} 5x\,dx.$$

$\Delta x = \dfrac{10-0}{4} = 2.5$. Here, $f(x_1) = f(2.5) = 12.5, f(x_2) = f(5) = 25,\ f(x_3) = f(7.5) = 37.5, f(x_4) = f(10) = 50$.

$$\int_0^{10} 5x\,dx = \frac{2.5}{2}(0 + 2(12.5 + 25 + 37.5) + 50) = 250.$$

5. **(c)**:

$$\int_0^{10} 5x\,dx = \frac{5}{2}x^2\Big|_0^{10} = 250.$$

11. **(a)**:

$\int_{-3}^{3} \sqrt{9-x^2}\,dx.$

$\Delta x = \frac{3-(-3)}{4} = 1.5$. Here, $f(x_0) = f(-3) = 0, f(x_1) = f(-1.5) = \frac{3\sqrt{3}}{2}$, $f(x_2) = f(0) = 9, f(x_3) = f(1.5) = \frac{3\sqrt{3}}{2}, f(x_4) = f(3) = 0.$

$\int_{-3}^{3} \sqrt{9-x^2}\,dx = \frac{1.5}{2}(0 + 2(\frac{3\sqrt{3}}{2} + 3 + \frac{3\sqrt{3}}{2}) + 0) = \frac{9}{2}(1+\sqrt{3}).$

11. **(c)**:

$\int_{-3}^{3} \sqrt{9-x^2}\,dx = \left(\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\arcsin\frac{x}{3}\right)\Big|_{-3}^{3} = \frac{9}{2}\pi.$

13. $\int_{-1}^{1} e^{x^2}\,dx.$

$\Delta x = \frac{1-(-1)}{6} = \frac{1}{3}$. $f(x_0) = f(-1) = ef(x_1) = f(-\frac{2}{3}) = e^{\frac{4}{9}}$, $f(x_2) = f(-\frac{1}{3}) = e^{\frac{1}{9}}, f(x_3) = f(0) = e^0 = 1, f(x_4) = f(\frac{1}{3}) = e^{\frac{1}{9}}$, $f(x_5) = f(\frac{2}{3}) = e^{\frac{4}{9}}, f(x_6) = f(1) = e^1 = e.$

$\int_{-1}^{1} e^{2^2}\,dx = \frac{1}{6}(e + 2(2e^{\frac{4}{9}} + 2e^{\frac{1}{9}} + 1) + e) = 3.0241.$

15. $\int_{0}^{\pi} x\sin x\,dx.$

$\Delta x = \frac{\pi - 0}{6} = \frac{\pi}{6}$. Here, $f(x_0) = f(0) = 0; f(x_1) = f(\frac{\pi}{6}) = \frac{\pi}{12}$, $f(x_2) = f(\frac{\pi}{3}) = \frac{\sqrt{3}\pi}{6}, f(x_3) = f(\frac{\pi}{2}) = \frac{\pi}{2}, f(x_4) = f(\frac{2\pi}{3}) = \frac{\pi}{\sqrt{3}}$, $f(x_5) = f(\frac{5\pi}{6}) = \frac{5\pi}{12}, f(x_6) = f(\pi) = 0.$

$\int_{0}^{\pi} x\sin x\,dx = \frac{\pi}{12}\Big(0 + 2\Big(\frac{\pi}{12} + \frac{\sqrt{3}\pi}{6} + \frac{\pi}{2} + \frac{\pi}{\sqrt{3}} + \frac{5\pi}{12}\Big) + 0\Big) = 3.0695.$

21. **(a)**:

$\int_{1}^{4} \frac{1}{\sqrt{x}}\,dx.$

$f" = \frac{3}{4}x^{-2.5} \qquad \frac{3}{128} < f" < \frac{3}{4} < 1 \qquad \Rightarrow M = 1.$

$E_T = \frac{(4-1)^3}{12n^2} < 0.0001 \qquad \Rightarrow n = 150.$

23. **(a)**:

$\int_{0}^{5} x^4\,dx.$

$f" = 12x^2 \qquad 0 < f" < 300 \qquad \Rightarrow M = 300.$

$E_T = 300\frac{(5-0)^3}{12n^2} < 0.0001 \qquad \Rightarrow n = 5591.$

Chapter 3

Applications of Integration

3.1 Area Between Curves

5. $A = \int_{-1}^{1} (-3x^3 + 3x + 2) - (x^2 + x - 1)\,dx = \int_{-1}^{1} (-3x^3 - x^2 + 2x + 3)\,dx.$

$$= \Big(-\frac{3}{4}x^4 - \frac{1}{3}x^3 + x^2 + 3x \Big)\Big|_{-1}^{1} = \frac{16}{3}.$$

7. $A = \int_{0}^{\pi} (\sin x + 1) - (\sin x)\,dx = \int_{0}^{\pi} 1\,dx = x\Big|_{0}^{\pi} = \pi.$

9. $A = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\,dx = (\cos x + \sin x)\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = 2\sqrt{2}.$

11. $2x^2 + 5x - 3 = x^2 + 4x - 1 \Rightarrow x^2 + x - 2 = 0 \Rightarrow x = 1, -2.$ Thus,

$$A = \int_{-2}^{1} (2x^2 + 5x - 3) - (x^2 + x - 2)\,dx = \int_{-2}^{1} (x^2 + x - 2)\,dx = \Big(\frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x \Big)\Big|_{-2}^{1} = \frac{9}{2}.$$

13. $\sin x - \frac{2x}{\pi} = 0 \Rightarrow x = -\frac{\pi}{2}, 0, \frac{\pi}{2}.$ Due to symmetry property, we take integral for the positive part (from 0 to $\frac{\pi}{2}$), and double the result.

$$A = 2\int_{0}^{\frac{\pi}{2}} \Big(\sin x - \frac{2x}{\pi} \Big)\,dx = 2\Big(-\cos x - \frac{x^2}{\pi} \Big)\Big|_{0}^{\frac{\pi}{2}} = 2 - \frac{\pi}{2}.$$

15. $x - \sqrt{x} = 0 \Rightarrow \sqrt{x}(\sqrt{x} - 1) = 0 \Rightarrow x = 0, 1.$

$$A = \int_{0}^{1} (\sqrt{x} - x)\,dx = \Big(\frac{2}{3}x^{1.5} - \frac{1}{2}x^2 \Big)\Big|_{0}^{1} = \frac{1}{6}.$$

19. $A = \int_{0}^{1} \Big(\sqrt{x} + \frac{1}{2}x \Big)\,dx + \int_{1}^{2} \Big((-2x + 3) + \frac{1}{2}x \Big)\,dx$

$$= \Big(\frac{2}{3}x^{1.5} + \frac{1}{4}x^2 \Big)\Big|_{0}^{1} + \Big(-x^2 + 3x + \frac{1}{4}x^2 \Big)\Big|_{1}^{2} = \frac{5}{3}.$$

21. $-\frac{1}{2}y+1=\frac{1}{2}y^2 \Rightarrow -y+2=y^2 \Rightarrow y^2+y-2=0 \Rightarrow y=-2,1$. So,

$$A=\int_{-1}^{1}\left(\frac{1}{2}y^2\right)dy+\int_{-2}^{1}\left(-\frac{1}{2}y+1-\frac{1}{2}y^2\right)dy=\frac{1}{6}y^3\Big|_{-1}^{1}+\left(-\frac{1}{4}y^2+y-\frac{1}{6}y^3\right)\Big|_{-2}^{1}=\frac{9}{4}.$$

23. The line goes through (1,1) and (2,3) is $y-1=\frac{3-1}{2-1}(x-1) \Rightarrow y=2(x-1)+1 \Rightarrow y=2x-1$.

The line goes through (1,1) and (3,3) is $y-1=\frac{3-1}{3-1}(x-1) \Rightarrow y=(x-1)+1 \Rightarrow y=x$.

The line goes through (2,3) and (3,3) is $y-3=\frac{3-3}{3-2}(x-3) \Rightarrow y=3$.

Intersection points are $(1.5,1.5)$, $(3,3)$ and $(2,3)$. Therefore,

$$A=\int_{1}^{2}\Big((2x-1)-x\Big)\,dx+\int_{2}^{3}(3-x)\,dx=\int_{1.5}^{2}(x-1)\,dx+\int_{2}^{3}(3-x)\,dx.$$

$$=\left(\frac{1}{2}x^2-x\right)\Big|_{1}^{2}+\left(3x-\frac{1}{2}x^2\right)\Big|_{2}^{3}=1.$$

Chapter 4

More Applications

4.1 Continuous Income Streams

1. $PV = \int_5^\infty 200 \cdot e^{-rt}\, dt = 200 \cdot \lim_{s\to\infty} \left(-\frac{1}{r}e^{-rt}\right)\Big|_5^s = 200 \cdot \lim_{s\to\infty} \frac{e^{-5r} - e^{-rs}}{r} = 200 \cdot \frac{e^{-5r}}{r}$.
$1000 \cdot \left[\frac{1 - e^{-5r}}{r}\right] = PV \Rightarrow 1000 \cdot \left[\frac{1 - e^{-5r}}{r}\right] = 200 \cdot \frac{e^{-5r}}{r} \Rightarrow 5 - 5e^{-5r} = e^{-5r} \Rightarrow 6e^{-5r} = 5 \Rightarrow e^{-5r} = \frac{5}{6} \Rightarrow e^{5r} = \frac{6}{5} \Rightarrow r = \frac{1}{5}\ln\left(\frac{6}{5}\right) \approx 3.64\%$.

3. The future value is $FV = \int_0^{40} f(t)\ dt = \int_0^{40} (2000 + 400t)e^{0.09(40-t)}dt = e^{3.6}\left[\frac{2000e^{-0.09t}}{-0.09}\right]_0^{40} + e^{3.6}\left[\frac{te^{-0.09t}}{-0.09} - \frac{e^{-0.09t}}{-0.09^2}\right]_0^{40} = 2,371,230$. Note that we use the fact that $\int e^{rt} = \frac{e^{rt}}{r} + c$ and $\int te^{rt} = \frac{te^{rt}}{r} - \frac{e^{rt}}{r^2} + c$ when evaluate this integral.
The answer is \$2,371,230, which in current dollars assuming a 4% rate of inflation is \$478,743 - enough to live on for a good number of years. The amount of principal deposited is $\int_0^{40} (2000{+}400t)dt = 400,000$, which leaves roughly two million of interest - the lion's share. Note that even the first \$2000 deposited becomes \$73,200.

5. The present value of saving is $\int_0^7 10,000e^{-0.10t}\ dt = \cdots = \$50,340$. The present value for maintenance and repair is $\int_0^7 (1000 + 200t)e^{-0.10t}\ dt = \cdots = \$8149.60$. The present value of the scrap for \$1000 in 7 years is $1000e^{-0.10\cdot 7} = \$496.60$. So, the maximum amount that we should be willing to pay now is $\$50,340 + \$8149.60 + \$496.60 = \$42,687$.

4.2 Consumers' and Producers' Surpluses

1. a) $D(x) = S(x) \Rightarrow 100 - 0.05x = 10 + 0.1x \Rightarrow 0.15x = 90 \Rightarrow x = 600 \Rightarrow p = 100 - 0.05(600) = 70$. So, the equilibrium quantity and price is $(70, 600)$.

   b) $CS = \int_0^{600} \Big[D(x) - 70\Big]\ dx = \int_0^{600} \Big[30 - 0.05x\Big]\ dx = 9000$ (dollars/unit), and
   $PS = \int_0^{600} \Big[70 - S(x)\Big]\ dx = \int_0^{600} \Big[60 - 0.1x\Big]\ dx = 18,000$ (dollars/unit)

3. a) $D(x) = S(x) \Rightarrow e^{9-x} = e^{x+3} \Rightarrow 9 - x = x + 3 \Rightarrow 2x = 6 \Rightarrow x = 3 \Rightarrow p = e^6 \approx 403$. So, the equilibrium quantity and price is $(403, 3)$.

   b) $CS = \int_0^3 \Big[D(x) - 403\Big]\,dx = \int_0^3 \Big[e^{9-x} - 403\Big]\,dx = 45,432$ (dollars/unit), and

   $PS = \int_0^3 \Big[403 - S(x)\Big]\,dx = \int_0^3 \Big[403 - e^{x+3}\Big]\,dx = ...$ (dollars/unit).

5. a) $D(x) = S(x) \Rightarrow 20e^{-x} = 5e^x \Rightarrow e^{2x} = 4 \Rightarrow 2x = \ln 4 = 2\ln 2 \Rightarrow x = \ln 2 \Rightarrow p = 5e^{\ln 2} = 10$. So, the equilibrium quantity and price is $(10, \ln 2)$.

   b) $CS = \int_0^{\ln 2} \Big[D(x) - 10\Big]\,dx = \int_0^{\ln 2} \Big[20e^{-x} - 10\Big]\,dx = 3.07$ (dollars/unit), and

   $PS = \int_0^{\ln 2} \Big[10 - S(x)\Big]\,dx = \int_0^{\ln 2} \Big[10 - 5e^x\Big]\,dx = 1.93$ (dollars/unit).

4.3 Probabilities

1. $PV = \int_5^\infty 200 \cdot e^{-rt}\,dt = 200 \cdot \lim_{s\to\infty} \Big(-\frac{1}{r}e^{-rt}\Big)\Big|_5^s = 200 \cdot \lim_{s\to\infty} \frac{e^{-5r} - e^{-rs}}{r} = 200 \cdot \frac{e^{-5r}}{r}$.

   $1000 \cdot \left[\frac{1 - e^{-5r}}{r}\right] = PV \Rightarrow 1000 \cdot \left[\frac{1 - e^{-5r}}{r}\right] = 200 \cdot \frac{e^{-5r}}{r} \Rightarrow 5 - 5e^{-5r} = e^{-5r} \Rightarrow 6e^{-5r} = 5 \Rightarrow e^{-5r} = \frac{5}{6} \Rightarrow e^{5r} = \frac{6}{5} \Rightarrow r = \frac{1}{5}\ln\left(\frac{6}{5}\right) \approx 3.64\%$.

3. $FV = \int_0^{40} f(t)\,dt = \int_0^{40} (2000 + 400t)e^{0.09(40-t)}dt = e^{3.6}\left[\frac{2000e^{-0.09t}}{-0.09}\right]_0^{40} + e^{3.6}\left[\frac{te^{-0.09t}}{-0.09} - \frac{e^{-0.09t}}{-0.09^2}\right]_0^{40}$ $= 2,371,230$. Note that we use the fact that $\int e^{rt} = \frac{e^{rt}}{r} + c$ and $\int te^{rt} = \frac{te^{rt}}{r} - \frac{e^{rt}}{r^2} + c$ when evaluate this integral.

   The answer is \$2,371,230, which in current dollars assuming a 4% rate of inflation is \$478,743 - enough to live on for a good number of years. The amount of principal deposited is $\int_0^{40} (2000 + 400t)dt = 400,000$, which leaves roughly two million of interest - the lion's share. Note that even the first \$2000 deposited becomes \$73,200.

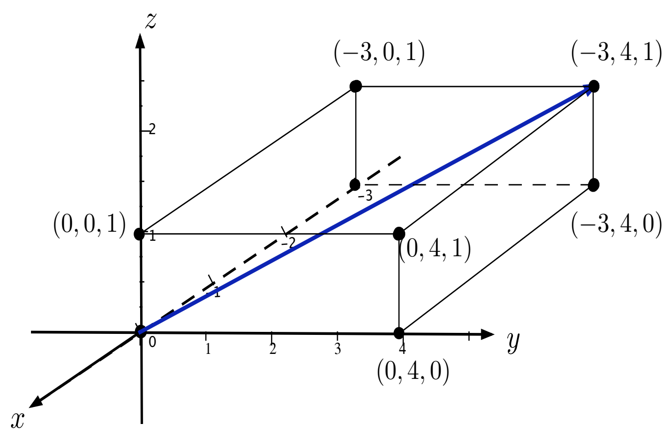
5. The present value of saving is $\int_0^7 10,000e^{-0.10t}\,dt = \cdots = \$50,340$. The present value for maintenance and repair is $\int_0^7 (1000 + 200t)e^{-0.10t}\,dt = \cdots = \$8149.60$. The present value of the scrap for \$1000 in 7 years is $1000e^{-0.10\cdot 7} = \$496.60$. So, the maximum amount that we should be willing to pay now is $\$50,340 + \$8149.60 + \$496.60 = \$42,687$.

Chapter 5

Multivariate Calculus

5.1 The Three-Dimensional Coordinate System

1. Position now $(-3,4,1)$; distance travelled is $3+4+1=8$ units; distance from origin $=\sqrt{(-3)^2+4^2+1^2}=\sqrt{26}\approx 5.0990$ units.



3. a) Midpoint $P_{1,2}$ between P_1 and $P_2=\left(\dfrac{1+7}{2},\dfrac{2-3}{2},\dfrac{1+4}{2}\right)=\left(4,-\dfrac{1}{2},\dfrac{5}{2}\right)$.

   b) Take $(1+6t,2-5t,1+3t)\Big|_{t=0}=(1,2,1)\equiv P_1$; take $(1+6t,2-5t,1+3t)\Big|_{t=1}=(1+6,2?5,1+3)=(7,-3,4)\equiv P_2$. Notice, $P_{1,2}=(1+6t,2-5t,1+3t)\Big|_{t=1/2}$.

   c) If $Q(2,7,-5)$ lies on the line through P_1 and P_2, then there is a common t-value that will give Q. Start with x-coordinate: $2=1+6t\Rightarrow t=1/6$; next y-coordinate: $7=2-5t\Rightarrow t=-1$. Different t-values for each coordinate means Q is not collinear with P_1 and P_2.

5. $x^2+y^2+z^2+x+y+z=0\Rightarrow x^2+x+y^2+y+z^2+z=0\Rightarrow x^2+2(1/2)x+(1/2)^2+y^2+2(1/2)y+(1/2)^2+z^2+2(1/2)z+(1/2)^2=3(1/2)^2\Rightarrow(x+1/2)^2+(y+1/2)^2+(z+1/2)^2=3/4$. Centre: $(-1/2,-1/2,-1/2)$; radius: $\sqrt{3}/2$.

7. Plane $x=8$: $64+y^2+z^2=100\Rightarrow y^2+z^2=36$. Circle in yz-plane with : centre $(0,0)$; radius: 6.

9. Plane $z=10$: $x^2+y^2+100=100\Rightarrow x^2+y^2=0$. Circle (point) in xy-plane with: centre $(0,0)$; radius: 0.

5.2 Planes and Surfaces

1. a) TRUE: planes' orientation is determined by their normals (relative size and sign of each variable's coordinate); so if two normal are each parallel to a third normal, then they must be parallel to each other; so the planes are parallel. Without normals terminology: if two planes are parallel, then the coordinates for each variable are in a fixed ratio (planes $x + y + z = d_1$ and $-77x - 77y - 77z = d_2$ are parallel); therefore, if if planes #1 and #2 are each parallel to plane #3, then the ratio of coordinates for the three variables of #1 to those of #2 will also be in a fixed ratio.

   b) FALSE: consider planes $x + y + z = d_1$ and $y = d_2$: they are both perpendicular to the plane $x - z = d_3$ but are not parallel to each other. Notice, however, that the converse is TRUE: two parallel planes will be perpendicular to the same planes.

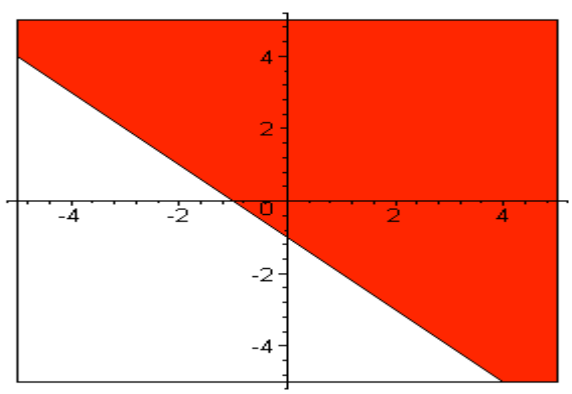
   c) FALSE: A plane parallel to a line means that the plane is parallel to a plane through the line (the plane can be moved - translated - so it contains the line); two non-parallel planes can certainly pass through the same line. For instance, planes $x + y + z = d_1$ and $y = d_2$ are both parallel to the line, passing through the origin, $x = z$, but as was argued in part (b), those planes are not parallel.

   d) TRUE: The quick answer is again using normal: the each plane's normal is perpendicular to that plane; so if two planes are perpendicular to one particular line, then those planes' normal must be parallel to the line, and so parallel to each other; and so the planes must be parallel.

3. The plane $5x - 13y + z = -7$ passes through all three points - you can check by substituting each point.

5. A point on the line of intersection of the planes must satisfy both planes' equations. That is, if the point is (p, q, r), then $p + q + r = 6$ and $p - q + r = 0$; subtracting these equations gives, $q = 3$ and so $p + r = 3$; pick two p-values, say $p = 0$ and $p = 1$; The associated points are $(0, 3, 3)$ and $(1, 3, 2)$.

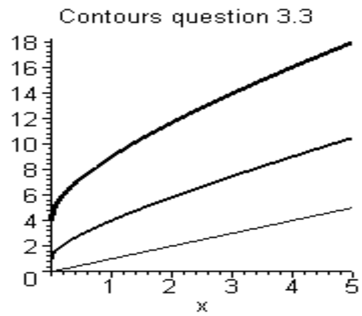
7. a) The domain of f is all (x, y) such that $1 + x + y > 0$; that is, the line $y = (1 + x)$ is the excluded boundary, and y-values should be larger than $(1 + x)$; as shown in figure below, where the feasible region is shaded and its diagonal edge is not included.



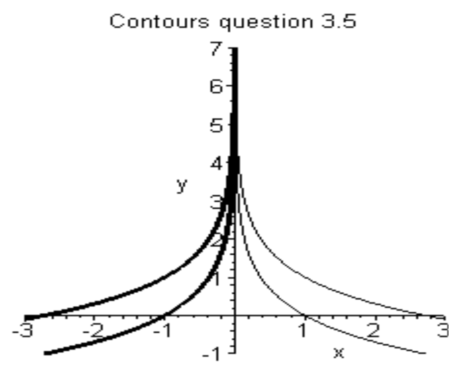
   b) The range of $\ln(1 + x + y)$ is $\mathbb{R}$, all real numbers.

5.3 Functions of Several Variables

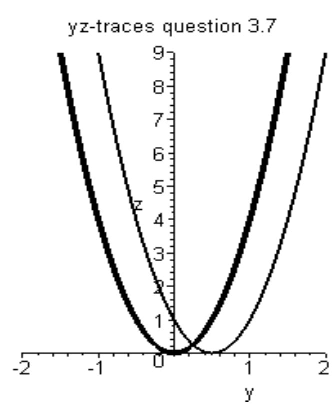
3. $z = y^{1/2} - x^{1/2} \Rightarrow y = (z + x)^2$, with $x, y > 0$. So for $z = 0, 1, 2$: $z = 0 \Rightarrow y = x, x > 0$; $z = 1 \Rightarrow y = \left(1 + x^{1/2}\right)^2$; and $z = 2 \Rightarrow y = \left(2 + x^{1/2}\right)^2$. The contours are in ascending order: the lowest (and thinnest) is for $z = 0$; the highest (and thickest) is for $z = 2$.



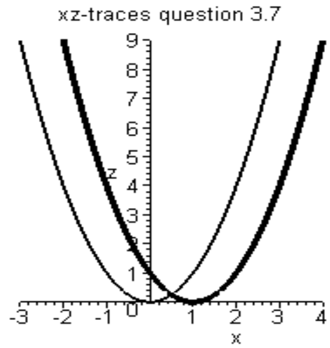
5. $z = xe^y \Rightarrow y = \ln(z/x) = \ln z - \ln x$, when x and z are positive, and $\ln(-z) - \ln(-x)$ when both are negative. For $z = 1 \Rightarrow y = -\ln x$; $z = -1 \Rightarrow y = -\ln(-x)$; $z = e \Rightarrow y = 1 - \ln x$; and $z = e \Rightarrow y = 1 - \ln(-x)$. The curves appear in symmetric pairs (reflected about the y-axis); the positive z-values, the first and third functions, are the thin curves on the right, positive side; the negative z-values, the second and fourth functions, are the thicker curves on the left, negative x-axis side; the lower curves are for the first two functions, with $|z| = 1$; and the upper curves are for the second two functions, with $|z| = e$.



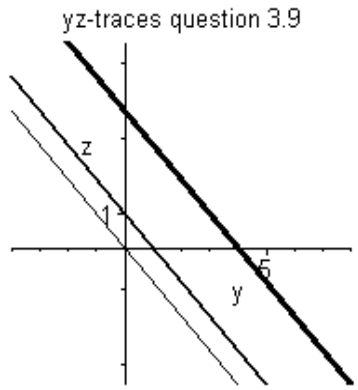
7. yz-traces: $x = 0 \Rightarrow z = (2y)^2 = 4y^2$; $x = 1 \Rightarrow z = (2y-1)^2$. These yz-plane graphs. .



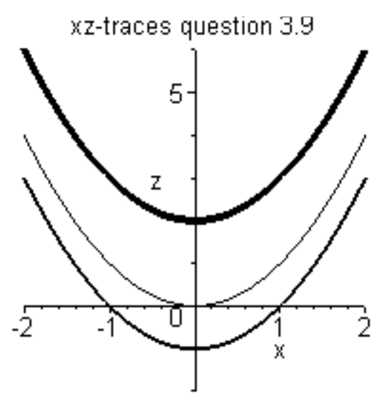
xz-traces: $y = 0 \Rightarrow z = (-x)^2 = x^2$, $y = 1/2 \Rightarrow z = (1-x)^2$. These xz-plane graphs.



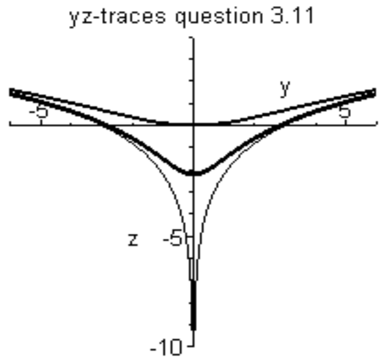
9. yz-traces: $x = 0 \Rightarrow z = -y$, the lower, thinnest line (passing through origin); $x = 1 \Rightarrow z = 1 - y$, middle line; $x = -2 \Rightarrow z = 4 - y$, upper, thickest line. These yz-plane graphs. .



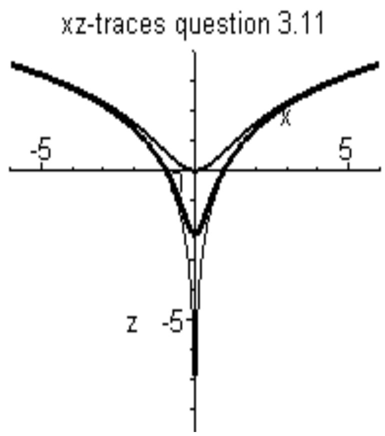
xz-traces: $y = 0 \Rightarrow z = x^2$, the middle, thinnest line (tangent to x-axis at origin); $y = 1 \Rightarrow z = x^2 - 1$, the lowest line; $y = -2 \Rightarrow z = x^2 + 2$, the top, thickest line (with z intercept 2). These xz-plane graphs.



11. yz-traces: $x = 0 \Rightarrow z = 2(\ln y - \ln 3)$, the lowest, thinnest line with the z-axis its vertical asymptote ; $x = 1 \Rightarrow z = \ln\left(1 + \frac{y^2}{9}\right)$, the top line tangent to the y-axis at the origin; $x = 1/3 \Rightarrow z = \ln(1 + y^2) - 2\ln 3$, the thickest line crossing the z-axis near $z = -2$. These yz-plane graphs. .
xz-traces: $y = 0 \Rightarrow z = 2\ln x$, the lowest, thinnest line with the z-axis its vertical asymptote ;



$y = 1 \Rightarrow z = \ln(x^2 + 1/9)$, the thickest line crossing the z-axis near $z = 2$; $y = 3 \Rightarrow z = \ln(x^2 + 1)$ is the highest line tangent to the x-axis at the origin. These xz-plane graphs.



5.4 Partial Derivatives

1. $f_x(x, y) = 100 \cdot \frac{2}{3} \cdot \left(\frac{y}{x}\right)^{1/3} \Rightarrow f_x(8, 27) = 100 \cdot \frac{2}{3} \cdot \left(\frac{27}{8}\right)^{1/3} = 100 \cdot \frac{2}{3} \cdot \frac{3}{2} = 100$. That is, the MPL is \$100.00 additional production for each additional hour labour used.

3. a) $f_x(0, 0) = 0$ [looks maybe < 0];
   b) $f_y(0, 0) = 0$;
   c) $f_x(-2, 0) < 0$;
   d) $f_y(-2, 0) > 0$ [looks maybe $= 0$].

5. $f_x = -72x;\ f_y = -128y;\ f_{xx} = -72;\ fxy = 0;\ f_{yy} = -128$. Thus, we have $D = (-72)(-128) - (0)^2 = 210 \cdot 32 = 9216 (> 0)$.

7. $f_x = \frac{y}{x},\ f_y = \ln x,\ f_{xx} = -\frac{y}{x^2},\ f_{xy} = \frac{1}{x},\ f_{yy} = 0$. Thus, we have $D = \Big(-\frac{y}{x^2}\Big) \cdot \Big(0\Big) - \Big(\frac{1}{x}\Big)^2 = -\frac{1}{x^2} < 0$.

9. $f_x = ye^{x+y},\ f_y = (1 + y)e^{x+y},\ f_{xx} = ye^{x+y},\ f_{xy} = (1 + y)e^{x+y},\ f_{yy} = (2 + y)e^{x+y}$. Thus, we have $D = \Big(ye^{x+y}\Big) \cdot \Big((2 + y)e^{x+y}\Big) - \Big((1 + y)e^{x+y}\Big)^2 = e^{2(x+y)}(-1) < 0$.

11. $f_x = \dfrac{y}{x^2+y^2}$, $f_y = \dfrac{-x}{x^2+y^2}$, $f_{xx} = \dfrac{-2xy}{(x^2+y^2)^2}$, $f_{xy} = \dfrac{x^2-y^2}{(x^2+y^2)^2}$, $f_{yy} = \dfrac{2xy}{(x^2+y^2)^2}$. Thus, we have $D = \Big(\dfrac{-2xy}{(x^2+y^2)^2}\Big)\cdot\Big(\dfrac{2xy}{(x^2+y^2)^2}\Big) - \Big(\dfrac{x^2-y^2}{(x^2+y^2)^2}\Big)^2 = -\dfrac{1}{(x^2+y^2)^2} < 0.$

13. $\dfrac{\partial z}{\partial x} = f'(xy)\cdot y \Rightarrow \dfrac{\partial z}{\partial x}\Big|_{1,1} = 5$, and $\dfrac{\partial z}{\partial y} = f'(xy)\cdot x \Rightarrow \dfrac{\partial z}{\partial y}\Big|_{1,1} = 5$. Now, $z_{xx} = f''(xy)\cdot y^2$, $z_{xy} = f'(xy) + f''(xy)\cdot xy$, $z_{yy} = f''(xy)\cdot x^2$. So, $D(x,y) = -\Big(f'(xy)\Big)^2 - 2xyf'(xy)f''(xy) \Rightarrow D(1,1) = 5.$

15. a) $f(x,y) = ax^{\alpha}y^{\beta} \Rightarrow f_x = a\alpha x^{\alpha-1}y^{\beta}$, $f_y = a\beta x^{\alpha}y^{\beta-1}$, $f_{xx} = a\alpha(\alpha-1)x^{\alpha-2}y^{\beta}$, $f_{xy} = a\alpha\beta x^{\alpha-1}y^{\beta-1}$, $f_{yy} = a\beta(\beta-1)x^{\alpha}y^{\beta-2}$.

b) $D(x,y) = \Big(a\alpha(\alpha-1)x^{\alpha-2}y^{\beta}\Big)\cdot\Big(a\beta(\beta-1)x^{\alpha}y^{\beta-2}\Big) - \Big(a\alpha\beta x^{\alpha-1}y^{\beta-1}\Big)^2 = a^2\alpha\beta x^{2(\alpha-1)}y^{2(\beta-1}\Big(1-\alpha-\beta\Big) = a^2x^{2(\alpha-1)}y^{2(\beta-1)}\{1-(\alpha+\beta)\}$. When, $\alpha+\beta = 1$, $D(x,y) = 0$ for all $(x,y)\in\mathbb{R}^2$.

5.5 Maxima and Minima

1.
- at $\Big(0,0,f(0,0)=0\Big)$, $T(x,y) = x+y$. $\Big(0,0,f(0,0)=0\Big)$, $T(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = x+y$.
- at $\Big(1,0,f(1,0)=\ln 2\Big)$, $T(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(0,0)(y-0) = \ln 2 - \dfrac{1}{2} + \dfrac{1}{2}x + \dfrac{1}{2}y$.
  at $\Big(1,0,f(1,0)=\ln 2\Big)$, $T(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(0,0)(y-0) = \ln 2 - \dfrac{1}{2} + \dfrac{1}{2}x + \dfrac{1}{2}y$.
- at $\Big(1,1,f(1,1) = \ln 3\Big)$, $T(x,y) = \ln 3 - \dfrac{2}{3} + \dfrac{1}{3}x + \dfrac{1}{3}y$. at $\Big(1,1,f(1,1) = \ln 3\Big)$, $T(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) = \ln 3 - \dfrac{2}{3} + \dfrac{1}{3}x + \dfrac{1}{3}y$.
- at $\Big(1,-1,f(1,-1) = 0\Big)$, $T(x,y) = x+y$. at $\Big(1,-1,f(1,-1) = 0\Big)$, $T(x,y) = f(1,-1) + f_x(1,-1)(x-1) + f_y(1,-1)(y+1) = x+y$.

3. $f_x = 2x+2$, $f_y = 2y-2$. So, at $\Big(1,1,f(1,1)=2\Big)$, we have $T(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) = 2 + 4(x-1) + 0(y-1) = -2+4x$. Thus, $f(0.9,1.1) \approx T(0.9,1.1) = -2+4(0.9) = 1.6$. $T(x,y) = -2+4x$. $f(0.9,1.1) \approx T(0.9,1.1) = -2+4(0.9) = 1.6$.

5. a) $D = (1)(1) - (-2)^2 = -3 < 0$. So, it is a saddle point.
   b) $D = (-1)(-2) - (1)^2 = 1 > 0$, $f_{xx} = -1 < 0$. So, it is a local maximum.
   c) $D = (-1)(1) - (0)^2 = -1 < 0$. So, it is a saddle point.

7. $f_x = 3x^2 + 2x = x(3x+2) = 0 \Rightarrow x = 0, -\dfrac{2}{3}$, $f_y = -3y^2 + 2y = y(-3y+2) = 0 \Rightarrow y = 0, \dfrac{2}{3}$. So, we have four points: $(0,0)$, $\Big(-\dfrac{2}{3},0\Big)$, $\Big(0,\dfrac{2}{3}\Big)$, $\Big(-\dfrac{2}{3},\dfrac{2}{3}\Big)$. Now, $f_{xx} = 6x+2$, $f_{xy} = 0$, $f_{yy} = -6y+2 \Rightarrow D(x,y) = (6x+2)(-6y+2) - (0)^2 = 4(3x+1)(1+3y)$. Then, we have
- $D(0,0) = 4 > 0$, $f_{xx}(0,0) = 2 > 0 \Rightarrow$ the point is a local minimum.
- $D\Big(-\dfrac{2}{3},0\Big) = -4 < 0 \Rightarrow$ the point is a saddle point.
- $D\Big(0,\dfrac{2}{3}\Big) = -4 < 0 \Rightarrow$ the point is a saddle point.
- $D\Big(-\dfrac{2}{3},\dfrac{2}{3}\Big) = 4 > 0$, $f_{xx}\Big(-\dfrac{2}{3},\dfrac{2}{3}\Big) < 0 \Rightarrow$ the point is a local maximum.

9. $f_x = -\dfrac{1}{x^2} + y = \dfrac{x^2y-1}{x^2} = 0 \Rightarrow y = \dfrac{1}{x^2},\ f_y = x + \dfrac{8}{y^2} = \dfrac{xy^2+8}{y^2} = 0 \Rightarrow xy^2 = -8$
$\Rightarrow x \cdot \dfrac{1}{x^2} = -8 \Rightarrow x^3 = -\dfrac{1}{8} \Rightarrow x = -\dfrac{1}{2} \Rightarrow y = 4$. So, we have a point: $\Big(-\dfrac{1}{2}, 4\Big)$.
Now, $f_{xx} = \dfrac{2}{x^3},\ f_{xy} = 1,\ f_{yy} = -\dfrac{16}{y^3} \Rightarrow D(x,y) = \Big(\dfrac{2}{x^3}\Big)\Big(-\dfrac{16}{y^3}\Big) - (1)^2 = -\dfrac{32}{x^3y^3} - 1$. Then, we have $D\Big(-\dfrac{1}{2}, 4\Big) = 3 > 0,\ f_{xx}\Big(-\dfrac{1}{2}, 4\Big) = -16 < 0 \Rightarrow$ the point is a local maximum.

11. $f_x = 3x^2 - 6y = 0 \Rightarrow x^2 = 2y,\ f_y = -6x + 3y^2 = 0 \Rightarrow 2x = y^2$. From here, we get $\Big(\dfrac{y^2}{2}\Big)^2 = 2y \Rightarrow$ $y^4 = 8y \Rightarrow y^4 - 8y = 0 \Rightarrow y(y^3 - 8) \Longrightarrow y = 0,\ y = 2 \Rightarrow x = 0,\ 2$ as well. So, we have two points: $(0,0),\ (2,2)$.
Now, $f_{xx} = 6x,\ f_{xy} = -6,\ f_{yy} = 6y \Rightarrow D(x,y) = (6x)(6y) - (-6)^2 = 36(xy-1)$. Then, we have

- $D(0,0) = -36 < 0 \Rightarrow$ the point is a saddle point.
- $D\Big(2,2\Big) = 108 > 0,\ f_{xx}\Big(2,2\Big) = 12 > 0 \Rightarrow$ the point is a local minimum.

13. $f_x = 6y - 2x = 0 \Rightarrow 3y = x,\ f_y = 6x - 18y^2 = 0 \Rightarrow x = 3y^2$. Solving these two equations gives $y = y^2 \Rightarrow y = 0,\ 1 \Rightarrow x = 0,\ 3$. So, we have two points: $(0,0),\ (3,1)$. Now, $f_{xx} = -2,\ f_{xy} = 6,\ f_{yy} = -36y \Rightarrow D(x,y) = (-2)(-36y) - (6)^2 = 36(2y-1)$. Then, we have

- $D(0,0) = -36 < 0 \Rightarrow$ the point is a saddle point.
- $D(3,1) = 180 > 0,\ f_{xx}(3,1) = -2 < 0 \Rightarrow$ the point is a local maximum.

5.6 Lagrange Multipliers

1. Here, $g = x^2 + y^2 - 1$. $f_x = \lambda g_x \Rightarrow 2x = \lambda(2x) \Rightarrow x = 0$ or $\lambda = 1$, but $x \neq 1$. $f_y = \lambda g_y \Rightarrow 1 = \lambda(2y)$. If $x = 0$ we have $y^2 = 1 \Rightarrow y = \pm 1$, and $\lambda = \pm\dfrac{1}{2}$. But, if $x \neq 0$, then $\lambda = 1 \Rightarrow y = \dfrac{1}{2} \Rightarrow x = \pm\dfrac{\sqrt{3}}{2}$.
So, we have four points: $(0,1),\ (0,-1),\ \Big(\pm\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\Big)$. Thus,

- $f(0,1) = 1$, corresponding to $\lambda = \dfrac{1}{2}$.
- $f(0,-1) = -1$, which is a minimum, corresponding to $\lambda = -\dfrac{1}{2}$.
- $f\Big(\pm\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\Big) = \dfrac{5}{4}$, which is a maximum, corresponding to $\lambda = 1$.

3. Here, $g = 3x + 5y - 47$. $f_x = \lambda g_x \Rightarrow 2(x-1) = 3\lambda$. $f_y = \lambda g_y \Rightarrow 2(y-2) = 5\lambda$. This gives $\lambda = \dfrac{2}{3}(x-1) = \dfrac{2}{3}(y-2) \Rightarrow -5x + 3y = 1$. Together with $3x + 5y = 47$, we can solve for x and y to be $x = 4,\ y = 7$. Thus, $f(4,7) = 30$, which is a minimum, corresponding to $\lambda = 2$.

5. a) $f\Big(\dfrac{1}{4}, \dfrac{1}{4}\Big) = \dfrac{1}{2}$, corresponding to $\lambda = 1$. Here, $g = \sqrt{x} + \sqrt{y} - 1$. $f_x = \lambda g_x \Rightarrow 1 = \lambda \cdot \dfrac{1}{2\sqrt{x}},\ x > 0$. $f_y = \lambda g_y \Rightarrow 2 = \lambda \cdot \dfrac{1}{2\sqrt{y}},\ y > 0$. This gives $\lambda = 2\sqrt{x} = 2\sqrt{y} \Rightarrow x = y$. Together with $\sqrt{x} + \sqrt{y} = 1$, we can solve for x and y to be $x = \dfrac{1}{4},\ y = \dfrac{1}{4}$. Thus, $f\Big(\dfrac{1}{4}, \dfrac{1}{4}\Big) = \dfrac{1}{2}$, corresponding to $\lambda = 1$.

b) $f\left(\frac{1}{4}, \frac{1}{4}\right) = \frac{1}{2}$, but $f(1,0) = f(0,1) = 1$. From $\sqrt{x} + \sqrt{y} = 1$, we can solve for y to be $y = (1 - \sqrt{x})^2 \Rightarrow y' = 2(1 - \sqrt{x}) \cdot \left(-\frac{1}{2\sqrt{x}}\right) = 1 - x^{-1/2} \Rightarrow y'' = \frac{1}{2}x^{-3/2} > 0$.

c) At this point, contour $z = \frac{1}{2}$ is a tangent to the constraint $\sqrt{x} + \sqrt{y} = 1$. Points $(0,1)$ and $(1,0)$ are corner point solutions (since $\sqrt{x} + \sqrt{y} = 1$ meets implicit constraint constraints $y \geq 0$ and $x \geq 0$.

Chapter 6

Differential Equations

6.1 Graphical and Numerical Solutions to Differential Equations

1. No Selected Questions.

6.2 Separable Differential Equations

5. $y' + 1 - y^2 = 0 \Rightarrow y' = y^2 - 1 \Rightarrow \dfrac{dy}{dx} = y^2 - 1.$

   $\Rightarrow \dfrac{dy}{y^2 - 1} = dx \Rightarrow \displaystyle\int \frac{1}{y^2-1}\,dy = \int dx \Rightarrow \int \frac{1}{(y-1)(y+1)}\,dy = \int dx.$

   $\Rightarrow \dfrac{1}{2}\displaystyle\int \frac{1}{(y-1)}\,dy - \frac{1}{2}\int \frac{1}{(y+1)}\,dy = \int dx.$

   $\Rightarrow \dfrac{1}{2}\ln|y-1| + \dfrac{1}{2}\ln|y+1| = x + C$

7. $xy' = 4y \Rightarrow \dfrac{dy}{dx} = \dfrac{4y}{x} \Rightarrow \dfrac{dy}{4y} = \dfrac{dx}{x} \Rightarrow \displaystyle\int \frac{1}{4y}\,dy = \int \frac{1}{x}\,dx.$

   $\Rightarrow \dfrac{1}{4}\ln|y| = \ln|x| + C$

9. $e^x yy' = e^{-y} + e^{-2x-y} \Rightarrow e^x yy' = e^{-y} + \dfrac{e^{-2x}}{e^y} \Rightarrow e^x yy'e^y = 1 + e^{-2x}.$

   $\Rightarrow yy'e^y = \dfrac{1 + e^{-2x}}{e^x} \Rightarrow ye^y dy = (e^{-x} + e^{-3x})dx.$

   $\Rightarrow ye^y - e^y = -e^{-x} - 3e^{-3x} + C.$

13. $y' = \dfrac{\sin x}{\cos y} \Rightarrow \dfrac{dy}{dx} = \dfrac{\sin x}{\cos y} \Rightarrow \cos y dy = \sin x dx \Rightarrow \displaystyle\int \cos y\,dy = \int \sin x\,dx$

    $\Rightarrow \sin y = -\cos x + C \Rightarrow \sin \dfrac{pi}{2} = \cos 0 + C \Rightarrow C = 2$

$\Rightarrow \sin y = -\cos x + 2.$

15. $y' = \dfrac{2x}{y + x^2 y} \Rightarrow \dfrac{dy}{dx} = \dfrac{2x}{y(x^2+1)} \Rightarrow ydy = \dfrac{2xdx}{x^2+1} \Rightarrow \int y\, dy = \int \dfrac{2x}{x^2+1}\, dx$

$\Rightarrow \frac{1}{2}y^2 = \ln(x^2+1) + C \Rightarrow \frac{1}{2}(-4)^2 = \ln(0+1) + C \Rightarrow C = 8.$

$\Rightarrow \frac{1}{2}y^2 - \ln(x^2+1) = 8.$

17. $y' = \dfrac{x\ln(x^2+1)}{y-1} \Rightarrow (y-1)dy = x\ln(x^2+1)dx \Rightarrow \int (y-1)\, dy = \int x\ln(x^2+1)\, dx.$

$\Rightarrow \frac{1}{2}y^2 - y = \frac{1}{2}\Big((x^2+1)\ln(x^2+1) - (x^2+1)\Big) + C.$

$\Rightarrow \frac{1}{2}2^2 - 2 = \frac{1}{2}((0+1)\ln(0+1) - (0+1)) + C \Rightarrow C = \frac{1}{2}.$

$\Rightarrow \frac{1}{2}y^2 - y = \frac{1}{2}\Big((x^2+1)\ln(x^2+1) - (x^2+1)\Big) + \frac{1}{2}.$

19. $y' = (\cos^2 x)(\cos^2 2y) \Rightarrow \dfrac{1}{\cos^2 2y}dy = (\cos^2 x)dx \Rightarrow \int \dfrac{1}{\cos^2 2y}\, dy = \int (\cos^2 x)\, dx.$

$\Rightarrow \frac{1}{2}\tan(2y) = \frac{1}{2}\int (\cos 2x + 1)\, dx = \frac{1}{4}\sin 2x + \frac{1}{2}x + C.$

$\Rightarrow \frac{1}{2}\tan 0 = \frac{1}{4}\sin 0 + \frac{1}{2}0 + C \Rightarrow C = 0.$

$\Rightarrow 2\tan 2y = 2x + \sin 2x.$

6.3 First Order Linear Differential Equations

1. $y' = 2y - 3 \Rightarrow \dfrac{dy}{2y-3} = dx \Rightarrow \int (2y-3)\, dy = \int\, dx \Rightarrow \frac{1}{2}\ln(2y-3) = x + C.$

$y' = 2y - 3 \Rightarrow \ln(2y-3) = 2x + C \Rightarrow 2y - 3 = Ce^{2x} \Rightarrow y = \frac{1}{2}(Ce^{2x} + 3).$

$\Rightarrow y = \frac{3}{2} + Ce^{2x}.$

3. $x^2y' - xy = 1 \Rightarrow y' - \dfrac{1}{x}y = \dfrac{1}{x^2} \Rightarrow \mu = e^{\int -\frac{1}{x}\, dx} \Rightarrow \mu = e^{-\ln x} = \dfrac{1}{x}.$

$\Rightarrow \dfrac{d}{dx}\Big(\dfrac{1}{x}y\Big) = \dfrac{1}{x}\dfrac{1}{x^2} \Rightarrow \dfrac{y}{x} = \int x^{-3}\, dx \Rightarrow \dfrac{y}{x} = -\dfrac{1}{2}x^{-2} + C \Rightarrow y = -\dfrac{1}{2x} + Cx.$

5. $(\cos^2 x \sin x)y' + (\cos^3 x)y = 1 \Rightarrow y' + \dfrac{\cos x}{\sin x}y = \dfrac{1}{\cos^2 x \sin x}$.

$\mu = e^{\int \frac{\cos x}{\sin x}\,dx} = e^{\ln|\sin x|} = |\sin x|$.

$\Rightarrow \dfrac{d}{dx}(y\sin x) = \dfrac{\sin x}{\cos^2 x \sin x} = \dfrac{1}{\cos^2 x} \Rightarrow y\sin x = \displaystyle\int \frac{1}{\cos^2 x}\,dx = \tan x + C.$

$\Rightarrow y = \sec x + C\csc x.$

7. $x^3y' - 3x^3y = x^4e^{2x} \Rightarrow y' - 3y = xe^{2x} \Rightarrow \mu = e^{-\int 3\,dx} = e^{-3x}$.

$\Rightarrow \dfrac{d}{dx}(ye^{-3x}) = xe^{-x} \Rightarrow ye^{-3x} = \displaystyle\int xe^{-x}\,dx \Rightarrow ye^{-3x} = -xe^{-x} - e^{-x} + C.$

$\Rightarrow y = -xe^{2x} - e^{2x} + Ce^{3x} = Ce^{3x} - (x+1)e^{2x}.$

9. $y' = y + 2xe^x \Rightarrow y' - y = 2xe^x \Rightarrow \mu = e^{\int -1\,dx} = e^{-x}$

$\Rightarrow \dfrac{d}{dx}(ye^{-x}) = 2x \Rightarrow ye^{-x} = \displaystyle\int 2x\,dx \Rightarrow ye^{-x} = x^2 + C \Rightarrow y = x^2e^x + Ce^x.$

$y(0) = 2 \Rightarrow 2 = 0e^0 + Ce^0 \Rightarrow C = 2 \Rightarrow y = x^2e^x + 2e^x.$

11. $xy' + (x+2)y = x \Rightarrow y' + \dfrac{x+2}{x}y = 1.$

$\Rightarrow \mu = e^{\int \frac{x+2}{x}\,dx} = e^{\int 1+\frac{2}{x}\,dx} = e^{x+2lnx} = x^2e^x.$

$\Rightarrow \dfrac{d}{dx}(yx^2e^x) = x^2e^x \Rightarrow yx^2e^x = \displaystyle\int x^2e^x\,dx \Rightarrow yx^2e^x = x^2e^x - \int 2xe^x\,dx.$

$\Rightarrow yx^2e^x = x^2e^x - 2\Big(xe^x - \displaystyle\int e^x\,dx\Big) \Rightarrow yx^2e^x = x^2e^x - 2\Big(xe^x - e^x\Big) + C.$

$\Rightarrow y = 1 - \dfrac{2}{x} + \dfrac{2}{x^2} + \dfrac{C}{x^2e^x}.$

$y(1) = 0 \Rightarrow 0 = 1 - 2 + 2 + \dfrac{C}{e} \Rightarrow C = -e \Rightarrow y = 1 - \dfrac{2}{x} + \dfrac{2}{x^2} - \dfrac{e}{x^2e^x}.$

13. $(x+1)y' + (x+2)y = 2xe^{-x} \Rightarrow y' + \dfrac{x+2}{x+1}y = \dfrac{2xe^{-x}}{x+1}.$

$\Rightarrow \mu = e^{\int \frac{x+2}{x+1}\,dx} = e^{x+\ln|x+1|} = (x+1)e^x.$

$\Rightarrow \dfrac{d}{dx}(y(x+1)e^x) = 2x \Rightarrow y(x+1)e^x = \displaystyle\int 2x\,dx = x^2 + C \Rightarrow y = \frac{x^2+C}{(x+1)e^x}.$

$$y(0)=1 \Rightarrow 1=\frac{C}{1} \Rightarrow C=1 \Rightarrow y=\frac{x^2+1}{(x+1)e^x}.$$

15. $(x^2-1)y'+2y=(x+1)^2 \Rightarrow y'+\frac{2}{x^2-1}y=\frac{x+1}{x-1}.$

$$\Rightarrow \mu=e^{\int \frac{2}{x^2-1}\,dx}=e^{\int \frac{2}{(x-1)(x+1)}\,dx}=e^{\int (\frac{1}{(x-1)}-\frac{1}{(x+1)})\,dx}=e^{\ln|x-1|-\ln|x+1|}=\frac{e^{\ln|x-1|}}{e^{\ln|x+1|}}.$$

$$=\frac{x-1}{x+1}.$$

$$\frac{d}{dx}\Big(y\frac{x-1}{x+1}\Big)=1 \Rightarrow y\Big(\frac{x-1}{x+1}\Big)=\int dx=x+C \Rightarrow y=\frac{(x+C)(x+1)}{x-1}.$$

$$y(0)=2 \Rightarrow 2=\frac{C}{-1} \Rightarrow C=-2 \Rightarrow y=\frac{(x-2)(x+1)}{x-1}.$$

6.4 Modeling with Differential Equations

1. $\frac{dy}{dx}=k(10-y) \Rightarrow \frac{dy}{10-y}=kdx \Rightarrow \int \frac{1}{10-y}\,dy=\int k\,dx.$

$$\Rightarrow -\ln(10-y)=kx+D \Rightarrow (10-y)^{-1}=e^{kx+D}.$$

$$\Rightarrow \frac{1}{10-y}=Ee^{kx} \text{ where } E=e^D \Rightarrow 10-y=\frac{1}{E}e^{-kx}.$$

$$\Rightarrow y=10-\frac{1}{E}e^{-kx} \Rightarrow y=10+Ce^{-kx}.$$

3. Let y be the number of students who heard the rumor at any time so (250-y) is the number of students who have not heard the rumor.

$$\frac{dy}{dt}=ky(250-y) \Rightarrow \frac{dy}{ky(250-y)}=dt \Rightarrow \int \frac{dy}{y(250-y)}\,dy=\int k\,dt.$$

$$\Rightarrow \frac{1}{250}\int \Big(\frac{1}{y}+\frac{1}{250-y}\Big)\,dy=\int k\,dt \Rightarrow \int \Big(\frac{1}{y}+\frac{1}{250-y}\Big)\,dy=\int 250k\,dt.$$

$$\Rightarrow \ln y-\ln(250-y)=250kt+D \Rightarrow \ln\Big(\frac{y}{250-y}\Big)=250kt+D.$$

$$y(0)=1 \Rightarrow \ln\frac{1}{240}=D \Rightarrow D=-\ln(240).$$

$$y(3)=75 \Rightarrow \ln\Big(\frac{75}{250-75}\Big)=750k-\ln(240).$$

$$\Rightarrow k = \frac{1}{750}\ln\left(\frac{75*240}{175}\right) = \frac{1}{750}\ln\left(\frac{720}{7}\right) = 6.18x10^{-3}.$$

$$\ln\left(\frac{y}{250-y}\right) = \frac{1}{3}\ln\left(\frac{720}{7}\right)t - \ln(240)$$

80 percents of the students have heard the rumor so $y = 200$.

$$\Rightarrow \ln\left(\frac{200}{250-200}\right) = \frac{1}{3}\ln\left(\frac{720}{7}\right)t - \ln(240) \Rightarrow t = 4.45 \text{ days.}$$

7. Let T denote the temperature of the object at any time t.

$$\Rightarrow \frac{dT}{dt} = k(T - (60 + 20e^{-0.25t})) \Rightarrow T' - kT = -k(60 + 20e^{-0.25t}).$$

$$\Rightarrow \mu = e^{\int -k\,dt} = e^{-kt} \Rightarrow d(Te^{-kt}) = -ke^{-kt}(60 + 20e^{-0.25t})dt.$$

$$\Rightarrow Te^{-kt} = \int -ke^{-kt}(60 + 20e^{-0.25t})\,dt = 60e^{-kt} + \frac{20}{k+0.25}e^{-t(k+0.25)} + C.$$

$$\Rightarrow T = 60 + \frac{20}{k+0.25}e^{-0.25t} + Ce^{kt}.$$

$T(0) = 100$ and $T(20) = 80 \Rightarrow T = 60 - 3.69858e^{-0.25t} + 43.69858e^{-0.0390169t}$.

11. Let y denote the amount of salt in the tank at anytime t.

Rate in $= 1*4 = 4$(g/min). To calculate rate out, we need to calculate the volume at anytime t, volume $= 1 + (4-3)t = 1 + t \Rightarrow$ Rate out $= \frac{y}{1+t} * 3 = \frac{3y}{1+t}$

$$\frac{dy}{dt} = 4 - \frac{3y}{1+t} \Rightarrow y' + \frac{3}{1+t}y = 4 \Rightarrow \mu = e^{\int \frac{3}{1+t}\,dt} = (1+t)^3.$$

$$\Rightarrow d(y(1+t)^3) = 4(1+t)^3 dt \Rightarrow y(1+t)^3 = \int 4(1+t)^3\,dt = (1+t)^4 + C.$$

$$\Rightarrow y = (1+t) + \frac{C}{(1+t)^3} \text{ We have } y(0) = 2 \Rightarrow 2 = 1 + C \Rightarrow C = 1.$$

$$\Rightarrow y = (1+t) + \frac{1}{(1+t)^3} \Rightarrow y(10) = 11.00075.$$