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Mathematics 1274

Calculus II - Economics & Business Applications

First Custom Edition 2022

This Book is Adapted from the Open Textbook Titled
APEX Calculus by Gregory Hartman

Supplemented with

Multivariate Calculus Notebook by

Pichmony Anhaouy

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Preface

This custom book is created in response to the ever increasing price of calculus textbooks available out there through many publication companies. It is created specifically for Mathematics 1274 at Langara College.

The first three chapters of this book are mainly taken from chapters 5, 6, and 7 of the open source calculus textbook titled APEX CALCULUS, Version 5.0, by Gregory HARTMAN, department of applied mathematics, Virginia Military Institute. The chapter on Differential Equations is authored by ROSS MAGI, available in chapter 20 of this version of the text. The complete calculus book is under the Creative Commons Attribution- NonCommercial-Share-Alike 4.0 International License. It is available for free at www.apexcalculus.com.

The chapters on more applications and multivariate calculus are taken from MATH 1274 LECTURE NOTES and the MULTIVARIATE CALCULUS NOTEBOOK, by Pichmony ANHAOUY, department of mathematics and statistics, Langara College. The exercises for each section of these chapters are taken from MATH 1274 PROBLEMS BOOK and EXERCISES MULTIVARIATE CALCULUS, by Ken COLLINS and Pichmony ANHAOUY.

I would like to thank my dear colleagues, particularly Ken COLLINS and Bruce AUBERTIN, and my former students in the Mathematics and Statistic Department for their encouragements and contributions that lead to the existence of this book.

For more information regarding this book, whether it is about contents or typos, please contact:

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1: Integration

1.1 Antiderivatives and Indefinite Integration

We have spent considerable time considering the derivatives of a function and their applications. In the following chapters, we are going to starting thinking in “the other direction.” That is, given a function $f(x)$, we are going to consider functions $F(x)$ such that $F'(x) = f(x)$. There are numerous reasons this will prove to be useful: these functions will help us compute areas, volumes, mass, force, pressure, work, and much more.

Given a function $y = f(x)$, a *differential equation* is one that incorporates y, x, and the derivatives of y. For instance, a simple differential equation is:

$$y' = 2x.$$

Solving a differential equation amounts to finding a function y that satisfies the given equation. Take a moment and consider that equation; can you find a function y such that $y' = 2x$?

Can you find another?

And yet another?

Hopefully one was able to come up with at least one solution: $y = x^2$. “Finding another” may have seemed impossible until one realizes that a function like $y = x^2 + 1$ also has a derivative of $2x$. Once that discovery is made, finding “yet another” is not difficult; the function $y = x^2 + 123,456,789$ also has a derivative of $2x$. The differential equation $y' = 2x$ has many solutions. This leads us to some definitions.

Definition 1.1.1 Antiderivatives and Indefinite Integrals

Let a function $f(x)$ be given. An **antiderivative** of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.

The set of all antiderivatives of $f(x)$ is the **indefinite integral of** f, denoted by

$$\int f(x)\ dx.$$

Make a note about our definition: we refer to *an* antiderivative of f, as opposed to *the* antiderivative of f, since there is *always* an infinite number of them. We often use upper-case letters to denote antiderivatives.

Knowing one antiderivative of f allows us to find infinitely more, simply by adding a constant. Not only does this give us *more* antiderivatives, it gives us *all* of them.

Theorem 1.1.1 Antiderivative Forms

Let $F(x)$ and $G(x)$ be antiderivatives of $f(x)$. Then there exists a constant C such that

$$G(x) = F(x) + C.$$

Given a function f and one of its antiderivatives F, we know *all* antiderivatives of f have the form $F(x) + C$ for some constant C. Using Definition 1.1.1, we can say that

$$\int f(x)\ dx = F(x) + C.$$

Let's analyze this indefinite integral notation.

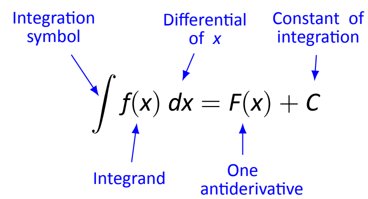


Figure 1.1: Understanding the indefinite integral notation.

Figure 1.1 shows the typical notation of the indefinite integral. The integration symbol, $\int$, is in reality an "elongated S," representing "take the sum." We will later see how *sums* and *antiderivatives* are related.

The function we want to find an antiderivative of is called the *integrand*. It contains the differential of the variable we are integrating with respect to. The $\int$ symbol and the differential dx are not "bookends" with a function sandwiched in between; rather, the symbol $\int$ means "find all antiderivatives of what follows," and the function $f(x)$ and dx are multiplied together; the dx does not "just sit there."

Let's practice using this notation.

Notes:

Example 1.1.1 Evaluating indefinite integrals

Evaluate $\int \sin x \; dx$.

Solution We are asked to find all functions $F(x)$ such that $F'(x) = \sin x$. Some thought will lead us to one solution: $F(x) = -\cos x$, because $\frac{d}{dx}(-\cos x) = \sin x$.

The indefinite integral of $\sin x$ is thus $-\cos x$, plus a constant of integration. So:

$$\int \sin x \; dx = -\cos x + C.$$

A commonly asked question is "What happened to the dx?" The unenlightened response is "Don't worry about it. It just goes away." A full understanding includes the following.

This process of *antidifferentiation* is really solving a *differential* question. The integral

$$\int \sin x \; dx$$

presents us with a differential, $dy = \sin x \; dx$. It is asking: "What is y?" We found lots of solutions, all of the form $y = -\cos x + C$.

Letting $dy = \sin x \; dx$, rewrite

$$\int \sin x \; dx \quad \text{as} \quad \int dy.$$

This is asking: "What functions have a differential of the form dy?" The answer is "Functions of the form $y + C$, where C is a constant." What is y? We have lots of choices, all differing by a constant; the simplest choice is $y = -\cos x$.

Understanding all of this is more important later as we try to find antiderivatives of more complicated functions. In this section, we will simply explore the rules of indefinite integration, and one can succeed for now with answering "What happened to the dx?" with "It went away."

Let's practice once more before stating integration rules.

Example 1.1.2 Evaluating indefinite integrals

Evaluate $\int (3x^2 + 4x + 5) \; dx$.

Solution We seek a function $F(x)$ whose derivative is $3x^2+4x+5$. When taking derivatives, we can consider functions term–by–term, so we can likely do that here.

Notes:

What functions have a derivative of $3x^2$? Some thought will lead us to a cubic, specifically $x^3 + C_1$, where C_1 is a constant.

What functions have a derivative of $4x$? Here the x term is raised to the first power, so we likely seek a quadratic. Some thought should lead us to $2x^2 + C_2$, where C_2 is a constant.

Finally, what functions have a derivative of 5? Functions of the form $5x + C_3$, where C_3 is a constant.

Our answer appears to be

$$\int (3x^2 + 4x + 5)\ dx = x^3 + C_1 + 2x^2 + C_2 + 5x + C_3.$$

We do not need three separate constants of integration; combine them as one constant, giving the final answer of

$$\int (3x^2 + 4x + 5)\ dx = x^3 + 2x^2 + 5x + C.$$

It is easy to verify our answer; take the derivative of $x^3 + 2x^3 + 5x + C$ and see we indeed get $3x^2 + 4x + 5$.

This final step of "verifying our answer" is important both practically and theoretically. In general, taking derivatives is easier than finding antiderivatives so checking our work is easy and vital as we learn.

We also see that taking the derivative of our answer returns the function in the integrand. Thus we can say that:

$$\frac{d}{dx}\left(\int f(x)\ dx\right) = f(x).$$

Differentiation "undoes" the work done by antidifferentiation.

We have seen a list of the derivatives of common functions we had learned at that point. We restate part of that list here to stress the relationship between derivatives and antiderivatives. This list will also be useful as a glossary of common antiderivatives as we learn.

Notes:

Theorem 1.1.2 Derivatives and Antiderivatives $(n \neq -1)$

Common Differentiation Rules

1. $\dfrac{d}{dx}\big(c_1 f(x) \pm c_2 g(x)\big) = c_1 f'(x) \pm c_2 g'(x)$
2. $\dfrac{d}{dx}\big(C\big) = 0$
3. $\dfrac{d}{dx}\big(x\big) = 1$
4. $\dfrac{d}{dx}\big(x^n\big) = n \cdot x^{n-1}$
5. $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = -\dfrac{1}{x^2}$
6. $\dfrac{d}{dx}\left(\sqrt{x}\right) = \dfrac{1}{2\sqrt{x}}$
7. $\dfrac{d}{dx}\big(\sin x\big) = \cos x$
8. $\dfrac{d}{dx}\big(\cos x\big) = -\sin x$
9. $\dfrac{d}{dx}\big(\tan x\big) = \sec^2 x$
10. $\dfrac{d}{dx}\big(\csc x\big) = -\csc x \cot x$
11. $\dfrac{d}{dx}\big(\sec x\big) = \sec x \tan x$
12. $\dfrac{d}{dx}\big(\cot x\big) = -\csc^2 x$
13. $\dfrac{d}{dx}\big(e^x\big) = e^x$
14. $\dfrac{d}{dx}\big(a^x\big) = \ln a \cdot a^x$
15. $\dfrac{d}{dx}\big(\ln x\big) = \dfrac{1}{x}$

Common Indefinite Integral Rules

1. $\displaystyle\int \big(c_1 f(x) \pm c_2 g(x)\big)\ dx = c_1 \int f(x)\ dx \pm c_2 \int g(x)\ dx$
2. $\displaystyle\int 0\ dx = C$
3. $\displaystyle\int 1\ dx = \int dx = x + C$
4. $\displaystyle\int x^n\ dx = \frac{1}{n+1} x^{n+1} + C$
5. $\displaystyle\int \frac{1}{x^2}\ dx = -\frac{1}{x} + C$
6. $\displaystyle\int \frac{1}{\sqrt{x}}\ dx = 2\sqrt{x} + C$
7. $\displaystyle\int \cos x\ dx = \sin x + C$
8. $\displaystyle\int \sin x\ dx = -\cos x + C$
9. $\displaystyle\int \sec^2 x\ dx = \tan x + C$
10. $\displaystyle\int \csc x \cot x\ dx = -\csc x + C$
11. $\displaystyle\int \sec x \tan x\ dx = \sec x + C$
12. $\displaystyle\int \csc^2 x\ dx = -\cot x + C$
13. $\displaystyle\int e^x\ dx = e^x + C$
14. $\displaystyle\int a^x\ dx = \frac{1}{\ln a} \cdot a^x + C$
15. $\displaystyle\int \frac{1}{x}\ dx = \ln|x| + C$

Notes:

We highlight a few important points from Theorem 1.1.2:

- Rule #1 states $\int c \cdot f(x)\ dx = c \cdot \int f(x)\ dx$. This is the Constant Multiple Rule: we can temporarily ignore constants when finding antiderivatives, just as we did when computing derivatives (i.e., $\frac{d}{dx}\big(3x^2\big)$ is just as easy to compute as $\frac{d}{dx}\big(x^2\big)$). An example:

$$\int 5\cos x\ dx = 5 \cdot \int \cos x\ dx = 5 \cdot (\sin x + C) = 5\sin x + C.$$

  In the last step we can consider the constant as also being multiplied by 5, but "5 times a constant" is still a constant, so we just write "C".

- Rule #2 is the Sum/Difference Rule: we can split integrals apart when the integrand contains terms that are added/subtracted, as we did in Example 1.1.2. So:

$$\begin{aligned}\int (3x^2 + 4x + 5)\ dx &= \int 3x^2\ dx + \int 4x\ dx + \int 5\ dx \\ &= 3\int x^2\ dx + 4\int x\ dx + \int 5\ dx \\ &= 3 \cdot \frac{1}{3}x^3 + 4 \cdot \frac{1}{2}x^2 + 5x + C \\ &= x^3 + 2x^2 + 5x + C\end{aligned}$$

  In practice we generally do not write out all these steps, but we demonstrate them here for completeness.

- Rule #5 is the Power Rule of indefinite integration. There are two important things to keep in mind:

  1. Notice the restriction that $n \neq -1$. This is important: $\int \frac{1}{x}\ dx \neq$ "$\frac{1}{0}x^0 + C$"; rather, see Rule #14.
  2. We are presenting antidifferentiation as the "inverse operation" of differentiation. Here is a useful quote to remember:

     > "Inverse operations do the opposite things in the opposite order."

     When taking a derivative using the Power Rule, we **first** *multiply* by the power, then **second** *subtract* 1 from the power. To find the antiderivative, do the opposite things in the opposite order: **first** *add* one to the power, then **second** *divide* by the power.

Notes:

- Note that Rule #14 incorporates the absolute value of x. The exercises will work the reader through why this is the case; for now, know the absolute value is important and cannot be ignored.

Initial Value Problems

In differential calculus we see that the derivative of a position function gave a velocity function, and the derivative of a velocity function describes acceleration. We can now go "the other way:" the antiderivative of an acceleration function gives a velocity function, etc. While there is just one derivative of a given function, there are infinite antiderivatives. Therefore we cannot ask "What is *the* velocity of an object whose acceleration is -32ft/s^2?", since there is more than one answer.

We can find *the* answer if we provide more information with the question, as done in the following example. Often the additional information comes in the form of an *initial value*, a value of the function that one knows beforehand.

Example 1.1.3 Solving initial value problems

The acceleration due to gravity of a falling object is -32 ft/s^2. At time $t = 3$, a falling object had a velocity of -10 ft/s. Find the equation of the object's velocity.

Solution We want to know a velocity function, $v(t)$. We know two things:

- The acceleration, i.e., $v'(t) = -32$, and
- the velocity at a specific time, i.e., $v(3) = -10$.

Using the first piece of information, we know that $v(t)$ is an antiderivative of $v'(t) = -32$. So we begin by finding the indefinite integral of -32:

$$\int (-32)\ dt = -32t + C = v(t).$$

Now we use the fact that $v(3) = -10$ to find C:

$$\begin{aligned} v(t) &= -32t + C \\ v(3) &= -10 \\ -32(3) + C &= -10 \\ C &= 86 \end{aligned}$$

Notes:

Thus $v(t) = -32t + 86$. We can use this equation to understand the motion of the object: when $t = 0$, the object had a velocity of $v(0) = 86$ ft/s. Since the velocity is positive, the object was moving upward.

When did the object begin moving down? Immediately after $v(t) = 0$:

$$-32t + 86 = 0 \quad \Rightarrow \quad t = \frac{43}{16} \approx 2.69\text{s}.$$

Recognize that we are able to determine quite a bit about the path of the object knowing just its acceleration and its velocity at a single point in time.

Example 1.1.4 Solving initial value problems

Find $f(t)$, given that $f''(t) = \cos t$, $f'(0) = 3$ and $f(0) = 5$.

Solution We start by finding $f'(t)$, which is an antiderivative of $f''(t)$:

$$\int f''(t)\ dt = \int \cos t\ dt = \sin t + C = f'(t).$$

So $f'(t) = \sin t + C$ for the correct value of C. We are given that $f'(0) = 3$, so:

$$f'(0) = 3 \quad \Rightarrow \quad \sin 0 + C = 3 \quad \Rightarrow \quad C = 3.$$

Using the initial value, we have found $f'(t) = \sin t + 3$.

We now find $f(t)$ by integrating again.

$$f(t) = \int f'(t)\ dt = \int (\sin t + 3)\ dt = -\cos t + 3t + C.$$

We are given that $f(0) = 5$, so

$$\begin{aligned} -\cos 0 + 3(0) + C &= 5 \\ -1 + C &= 5 \\ C &= 6 \end{aligned}$$

Thus $f(t) = -\cos t + 3t + 6$.

This section introduced antiderivatives and the indefinite integral. We found they are needed when finding a function given information about its derivative(s). For instance, we found a position function given a velocity function.

In the next section, we will see how position and velocity are unexpectedly related by the areas of certain regions on a graph of the velocity function. Then, in Section 1.4, we will see how areas and antiderivatives are closely tied together.

Notes:

Exercises 1.1

Terms and Concepts

1. Define the term "antiderivative" in your own words.

2. Is it more accurate to refer to "the" antiderivative of $f(x)$ or "an" antiderivative of $f(x)$?

3. Use your own words to define the indefinite integral of $f(x)$.

4. Fill in the blanks: "Inverse operations do the ________ things in the ________ order."

5. What is an "initial value problem"?

6. The derivative of a position function is a ________ function.

7. The antiderivative of an acceleration function is a ________ function.

Problems

In Exercises 8 – 26, evaluate the given indefinite integral.

8. $\displaystyle\int 3x^3\ dx$

9. $\displaystyle\int x^8\ dx$

10. $\displaystyle\int (10x^2 - 2)\ dx$

11. $\displaystyle\int dt$

12. $\displaystyle\int 1\ ds$

13. $\displaystyle\int \frac{1}{3t^2}\ dt$

14. $\displaystyle\int \frac{3}{t^2}\ dt$

15. $\displaystyle\int \frac{1}{\sqrt{x}}\ dx$

16. $\displaystyle\int \sec^2\theta\ d\theta$

17. $\displaystyle\int \sin\theta\ d\theta$

18. $\displaystyle\int (\sec x\tan x + \csc x\cot x)\ dx$

19. $\displaystyle\int 5e^\theta\ d\theta$

20. $\displaystyle\int 3^t\ dt$

21. $\displaystyle\int \frac{5^t}{2}\ dt$

22. $\displaystyle\int (2t+3)^2\ dt$

23. $\displaystyle\int (t^2+3)(t^3-2t)\ dt$

24. $\displaystyle\int x^2x^3\ dx$

25. $\displaystyle\int e^\pi\ dx$

26. $\displaystyle\int a\ dx$

27. This problem investigates why Theorem 1.1.2 states that $\displaystyle\int \frac{1}{x}\ dx = \ln|x| + C$.
    (a) What is the domain of $y = \ln x$?
    (b) Find $\frac{d}{dx}\big(\ln x\big)$.
    (c) What is the domain of $y = \ln(-x)$?
    (d) Find $\frac{d}{dx}\big(\ln(-x)\big)$.
    (e) You should find that $1/x$ has two types of antiderivatives, depending on whether $x > 0$ or $x < 0$. In one expression, give a formula for $\displaystyle\int \frac{1}{x}\ dx$ that takes these different domains into account, and explain your answer.

In Exercises 28 – 38, find $f(x)$ described by the given initial value problem.

28. $f'(x) = \sin x$ and $f(0) = 2$

29. $f'(x) = 5e^x$ and $f(0) = 10$

30. $f'(x) = 4x^3 - 3x^2$ and $f(-1) = 9$

31. $f'(x) = \sec^2 x$ and $f(\pi/4) = 5$

32. $f'(x) = 7^x$ and $f(2) = 1$

33. $f''(x) = 5$ and $f'(0) = 7,\ f(0) = 3$

34. $f''(x) = 7x$ and $f'(1) = -1,\ f(1) = 10$

35. $f''(x) = 5e^x$ and $f'(0) = 3,\ f(0) = 5$

36. $f''(\theta) = \sin\theta$ and $f'(\pi) = 2,\ f(\pi) = 4$

37. $f''(x) = 24x^2 + 2^x - \cos x$ and $f'(0) = 5,\ f(0) = 0$

38. $f''(x) = 0$ and $f'(1) = 3,\ f(1) = 1$

Review

39. Use information gained from the first and second derivatives to sketch $f(x) = \dfrac{1}{e^x + 1}$.

40. Given $y = x^2 e^x \cos x$, find dy.

1.2 The Definite Integral

We start with an easy problem. An object travels in a straight line at a constant velocity of 5 ft/s for 10 seconds. How far away from its starting point is the object?

We approach this problem with the familiar "Distance = Rate × Time" equation. In this case, Distance = 5ft/s × 10s = 50 feet.

It is interesting to note that this solution of 50 feet can be represented graphically. Consider Figure 1.2, where the constant velocity of 5ft/s is graphed on the axes. Shading the area under the line from $t = 0$ to $t = 10$ gives a rectangle with an area of 50 square units; when one considers the units of the axes, we can say this area represents 50 ft.

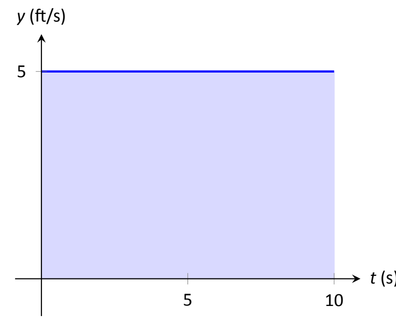


Figure 1.2: The area under a constant velocity function corresponds to distance traveled.

Now consider a slightly harder situation (and not particularly realistic): an object travels in a straight line with a constant velocity of 5ft/s for 10 seconds, then instantly reverses course at a rate of 2ft/s for 4 seconds. (Since the object is traveling in the opposite direction when reversing course, we say the velocity is a constant −2ft/s.) How far away from the starting point is the object – what is its *displacement*?

Here we use "Distance = $\text{Rate}_1 \times \text{Time}_1 + \text{Rate}_2 \times \text{Time}_2$," which is

$$\text{Distance} \quad = 5 \cdot 10 + (-2) \cdot 4 = 42 \text{ ft.}$$

Hence the object is 42 feet from its starting location.

We can again depict this situation graphically. In Figure 1.3 we have the velocities graphed as straight lines on $[0, 10]$ and $[10, 14]$, respectively. The displacement of the object is

$$\text{“Area above the } t\text{–axis} \quad - \quad \text{Area below the } t\text{–axis,”}$$

which is easy to calculate as $50 - 8 = 42$ feet.

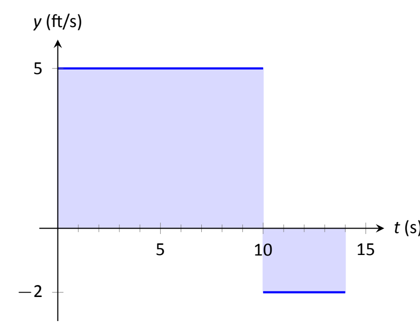


Figure 1.3: The total displacement is the area above the t–axis minus the area below the t–axis.

Now consider a more difficult problem.

Example 1.2.1 Finding position using velocity

The velocity of an object moving straight up/down under the acceleration of gravity is given as $v(t) = -32t + 48$, where time t is given in seconds and velocity is in ft/s. When $t = 0$, the object had a height of 0 ft.

1. What was the initial velocity of the object?
2. What was the maximum height of the object?
3. What was the height of the object at time $t = 2$?

Solution It is straightforward to find the initial velocity; at time $t = 0$, $v(0) = -32 \cdot 0 + 48 = 48$ ft/s.

Notes:

To answer questions about the height of the object, we need to find the object's position function $s(t)$. This is an initial value problem, which we studied in the previous section. We are told the initial height is 0, i.e., $s(0) = 0$. We know $s'(t) = v(t) = -32t + 48$. To find s, we find the indefinite integral of $v(t)$:

$$\int v(t)\ dt = \int (-32t + 48)\ dt = -16t^2 + 48t + C = s(t).$$

Since $s(0) = 0$, we conclude that $C = 0$ and $s(t) = -16t^2 + 48t$.

To find the maximum height of the object, we need to find the maximum of s. Recalling our work finding extreme values, we find the critical points of s by setting its derivative equal to 0 and solving for t:

$$s'(t) = -32t + 48 = 0 \quad \Rightarrow \quad t = 48/32 = 1.5\text{s}.$$

(Notice how we ended up just finding when the velocity was 0ft/s!) The first derivative test shows this is a maximum, so the maximum height of the object is found at

$$s(1.5) = -16(1.5)^2 + 48(1.5) = 36\text{ft}.$$

The height at time $t = 2$ is now straightforward to compute: it is $s(2) =$ 32ft.

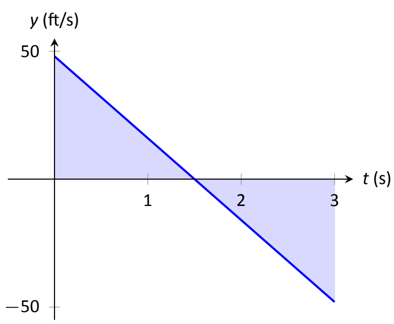


Figure 1.4: A graph of $v(t) = -32t + 48$; the shaded areas help determine displacement.

While we have answered all three questions, let's look at them again graphically, using the concepts of area that we explored earlier.

Figure 1.4 shows a graph of $v(t)$ on axes from $t = 0$ to $t = 3$. It is again straightforward to find $v(0)$. How can we use the graph to find the maximum height of the object?

Recall how in our previous work that the displacement of the object (in this case, its height) was found as the area under the velocity curve, as shaded in the figure. Moreover, the area between the curve and the t–axis that is below the t–axis counted as "negative" area. That is, it represents the object coming back toward its starting position. So to find the maximum distance from the starting point − the maximum height − we find the area under the velocity line that is above the t–axis, i.e., from $t = 0$ to $t = 1.5$. This region is a triangle; its area is

$$\text{Area } = \frac{1}{2}\text{Base} \times \text{Height} = \frac{1}{2} \times 1.5\text{s} \times 48\text{ft/s} = 36\text{ft},$$

which matches our previous calculation of the maximum height.

Finally, to find the height of the object at time $t = 2$ we calculate the total signed area under the velocity function from $t = 0$ to $t = 2$. This

Notes:

signed area is equal to $s(2)$, the displacement (i.e., signed distance) from the starting position at $t = 0$ to the position at time $t = 2$. That is,

$$\text{Displacement} = \text{Area above the } t\text{–axis} - \text{Area below } t\text{–axis}.$$

The regions are triangles, and we find

$$\text{Displacement} = \frac{1}{2}(1.5\text{s})(48\text{ft/s}) - \frac{1}{2}(.5\text{s})(16\text{ft/s}) = 32\text{ft}.$$

This also matches our previous calculation of the height at $t = 2$.

Notice how we answered each question in this example in two ways. Our first method was to manipulate equations using our understanding of antiderivatives and derivatives. Our second method was geometric: we answered questions looking at a graph and finding the areas of certain regions of this graph.

The above example does not *prove* a relationship between area under a velocity function and displacement, but it does imply a relationship exists. Section 1.4 will fully establish fact that the area under a velocity function is displacement.

Given a graph of a function $y = f(x)$, we will find that there is great use in computing the area between the curve $y = f(x)$ and the x-axis. Because of this, we need to define some terms.

Definition 1.2.1 The Definite Integral, Total Signed Area

Let $y = f(x)$ be defined on a closed interval $[a, b]$. The **total signed area from $x = a$ to $x = b$ under f** is:
(area under f and above the x–axis on $[a, b]$) − (area above f and under the x–axis on $[a, b]$).

The **definite integral of f on $[a, b]$** is the total signed area of f on $[a, b]$, denoted

$$\int_a^b f(x)\, dx,$$

where a and b are the **bounds of integration.**

By our definition, the definite integral gives the "signed area under f." We usually drop the word "signed" when talking about the definite integral, and simply say the definite integral gives "the area under f" or, more commonly, "the area under the curve."

Notes:

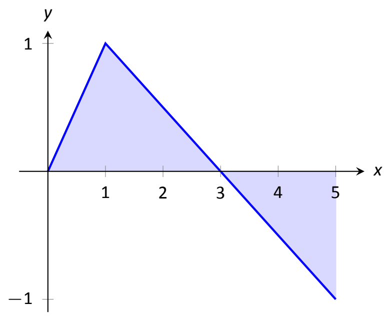


Figure 1.5: A graph of $f(x)$ in Example 1.2.2.

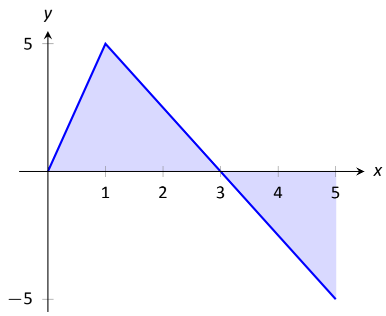


Figure 1.6: A graph of $5f$ in Example 1.2.2. (Yes, it looks just like the graph of f in Figure 1.5, just with a different y-scale.)

The previous section introduced the indefinite integral, which related to antiderivatives. We have now defined the definite integral, which relates to areas under a function. The two are very much related, as we'll see when we learn the Fundamental Theorem of Calculus in Section 1.4. Recall that earlier we said that the "$\int$" symbol was an "elongated S" that represented finding a "sum." In the context of the definite integral, this notation makes a bit more sense, as we are adding up areas under the function f.

We practice using this notation.

Example 1.2.2 Evaluating definite integrals

Consider the function f given in Figure 1.5.

Find:

1. $\displaystyle\int_0^3 f(x)\ dx$
2. $\displaystyle\int_3^5 f(x)\ dx$
3. $\displaystyle\int_0^5 f(x)\ dx$
4. $\displaystyle\int_0^3 5f(x)\ dx$
5. $\displaystyle\int_1^1 f(x)\ dx$

Solution

1. $\int_0^3 f(x)\ dx$ is the area under f on the interval $[0, 3]$. This region is a triangle, so the area is $\int_0^3 f(x)\ dx = \frac{1}{2}(3)(1) = 1.5$.

2. $\int_3^5 f(x)\ dx$ represents the area of the triangle found under the x–axis on $[3, 5]$. The area is $\frac{1}{2}(2)(1) = 1$; since it is found *under* the x–axis, this is "negative area." Therefore $\int_3^5 f(x)\ dx = -1$.

3. $\int_0^5 f(x)\ dx$ is the total signed area under f on $[0, 5]$. This is $1.5 + (-1) = 0.5$.

4. $\int_0^3 5f(x)\ dx$ is the area under $5f$ on $[0, 3]$. This is sketched in Figure 1.6. Again, the region is a triangle, with height 5 times that of the height of the original triangle. Thus the area is $\int_0^3 5f(x)\ dx = 15/2 = 7.5$.

5. $\int_1^1 f(x)\ dx$ is the area under f on the "interval" $[1, 1]$. This describes a line segment, not a region; it has no width. Therefore the area is 0.

Notes:

This example illustrates some of the properties of the definite integral, given here.

Theorem 1.2.1 Properties of the Definite Integral

Let f and g be defined on a closed interval I that contains the values a, b and c, and let k be a constant. The following hold:

1. $\displaystyle\int_a^a f(x)\,dx = 0$
2. $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
3. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
4. $\displaystyle\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. $\displaystyle\int_a^b k\cdot f(x)\,dx = k\cdot\int_a^b f(x)\,dx$

We give a brief justification of Theorem 1.2.1 here.

1. As demonstrated in Example 1.2.2, there is no "area under the curve" when the region has no width; hence this definite integral is 0.

2. This states that total area is the sum of the areas of subregions. It is easily considered when we let $a < b < c$. We can break the interval $[a, c]$ into two subintervals, $[a, b]$ and $[b, c]$. The total area over $[a, c]$ is the area over $[a, b]$ plus the area over $[b, c]$.

   It is important to note that this still holds true even if $a < b < c$ is not true. We discuss this in the next point.

3. This property can be viewed a merely a convention to make other properties work well. (Later we will see how this property has a justification all its own, not necessarily in support of other properties.) Suppose $b < a < c$. The discussion from the previous point clearly

Notes:

justifies

$$\int_b^a f(x)\ dx + \int_a^c f(x)\ dx = \int_b^c f(x)\ dx. \tag{1.1}$$

However, we still claim that, as originally stated,

$$\int_a^b f(x)\ dx + \int_b^c f(x)\ dx = \int_a^c f(x)\ dx. \tag{1.2}$$

How do Equations (1.1) and (1.2) relate? Start with Equation (1.1):

$$\begin{aligned} \int_b^a f(x)\ dx + \int_a^c f(x)\ dx &= \int_b^c f(x)\ dx \\ \int_a^c f(x)\ dx &= -\int_b^a f(x)\ dx + \int_b^c f(x)\ dx \end{aligned}$$

Property (3) justifies changing the sign and switching the bounds of integration on the $-\int_b^a f(x)\ dx$ term; when this is done, Equations (1.1) and (1.2) are equivalent.

The conclusion is this: by adopting the convention of Property (3), Property (2) holds no matter the order of a, b and c. Again, in the next section we will see another justification for this property.

4,5. Each of these may be non–intuitive. Property (5) states that when one scales a function by, for instance, 7, the area of the enclosed region also is scaled by a factor of 7. Both Properties (4) and (5) can be proved using geometry. The details are not complicated but are not discussed here.

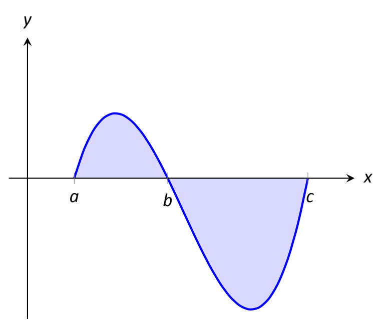


Figure 1.7: A graph of a function in Example 1.2.3.

Example 1.2.3 Evaluating definite integrals using Theorem 1.2.1.
Consider the graph of a function $f(x)$ shown in Figure 1.7. Answer the following:

1. Which value is greater: $\int_a^b f(x)\ dx$ or $\int_b^c f(x)\ dx$?
2. Is $\int_a^c f(x)\ dx$ greater or less than 0?
3. Which value is greater: $\int_a^b f(x)\ dx$ or $\int_c^b f(x)\ dx$?

Solution

Notes:

1. $\int_a^b f(x)\,dx$ has a positive value (since the area is above the x–axis) whereas $\int_b^c f(x)\,dx$ has a negative value. Hence $\int_a^b f(x)\,dx$ is bigger.

2. $\int_a^c f(x)\,dx$ is the total signed area under f between $x = a$ and $x = c$. Since the region below the x–axis looks to be larger than the region above, we conclude that the definite integral has a value less than 0.

3. Note how the second integral has the bounds "reversed." Therefore $\int_c^b f(x)\,dx$ represents a positive number, greater than the area described by the first definite integral. Hence $\int_c^b f(x)\,dx$ is greater.

The area definition of the definite integral allows us to use geometry compute the definite integral of some simple functions.

Example 1.2.4 Evaluating definite integrals using geometry

Evaluate the following definite integrals:

$$1.\ \int_{-2}^{5} (2x-4)\,dx \qquad 2.\ \int_{-3}^{3} \sqrt{9-x^2}\,dx.$$

Solution

1. It is useful to sketch the function in the integrand, as shown in Figure 1.8(a). We see we need to compute the areas of two regions, which we have labeled R_1 and R_2. Both are triangles, so the area computation is straightforward:

$$R_1 : \frac{1}{2}(4)(8) = 16 \qquad R_2 : \frac{1}{2}(3)6 = 9.$$

Region R_1 lies under the x–axis, hence it is counted as negative area (we can think of the triangle's height as being "-8"), so

$$\int_{-2}^{5} (2x-4)\,dx = -16 + 9 = -7.$$

2. Recognize that the integrand of this definite integral describes a half circle, as sketched in Figure 1.8(b), with radius 3. Thus the area is:

$$\int_{-3}^{3} \sqrt{9-x^2}\,dx = \frac{1}{2}\pi r^2 = \frac{9}{2}\pi.$$

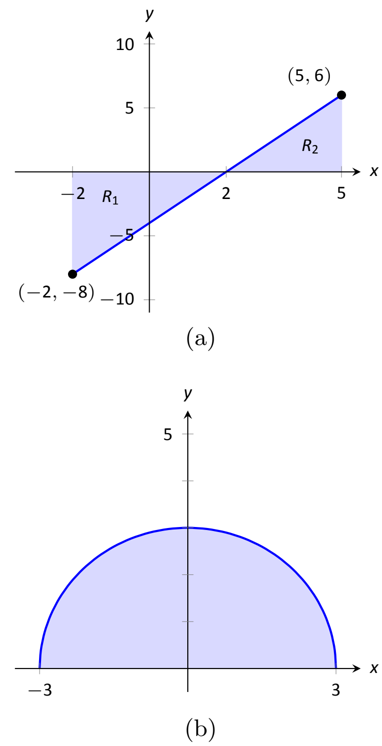


Figure 1.8: A graph of $f(x) = 2x - 4$ in (a) and $f(x) = \sqrt{9-x^2}$ in (b), from Example 1.2.4.

Notes:

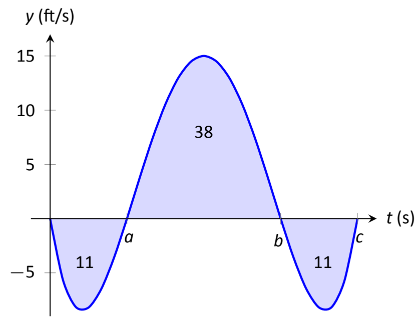


Figure 1.9: A graph of a velocity in Example 1.2.5.

Example 1.2.5 Understanding motion given velocity

Consider the graph of a velocity function of an object moving in a straight line, given in Figure 1.9, where the numbers in the given regions gives the area of that region. Assume that the definite integral of a velocity function gives displacement. Find the maximum speed of the object and its maximum displacement from its starting position.

Solution Since the graph gives velocity, finding the maximum speed is simple: it looks to be 15ft/s.

At time $t = 0$, the displacement is 0; the object is at its starting position. At time $t = a$, the object has moved backward 11 feet. Between times $t = a$ and $t = b$, the object moves forward 38 feet, bringing it into a position 27 feet forward of its starting position. From $t = b$ to $t = c$ the object is moving backwards again, hence its maximum displacement is 27 feet from its starting position.

In our examples, we have either found the areas of regions that have nice geometric shapes (such as rectangles, triangles and circles) or the areas were given to us. Consider Figure 1.10, where a region below $y = x^2$ is shaded. What is its area? The function $y = x^2$ is relatively simple, yet the shape it defines has an area that is not simple to find geometrically.

In the next section we will explore how to find the areas of such regions.

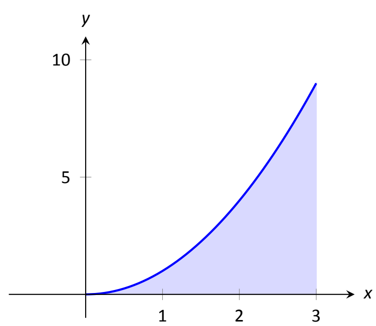


Figure 1.10: What is the area below $y = x^2$ on $[0, 3]$? The region is not a usual geometric shape.

Notes:

Exercises 1.2

Terms and Concepts

1. What is "total signed area"?

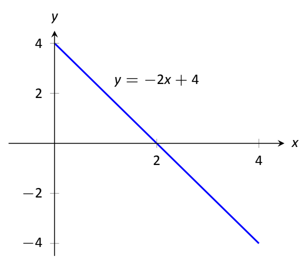
2. What is "displacement"?

3. What is $\displaystyle\int_3^3 \sin x\ dx$?

4. Give a single definite integral that has the same value as
$\displaystyle\int_0^1 (2x+3)\ dx + \int_1^2 (2x+3)\ dx.$

Problems

In Exercises 5 – 9, a graph of a function $f(x)$ is given. Using the geometry of the graph, evaluate the definite integrals.

5. 

(a) $\displaystyle\int_0^1 (-2x+4)\ dx$

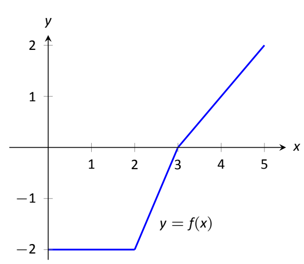
(b) $\displaystyle\int_0^2 (-2x+4)\ dx$

(c) $\displaystyle\int_0^3 (-2x+4)\ dx$

(d) $\displaystyle\int_1^3 (-2x+4)\ dx$

(e) $\displaystyle\int_2^4 (-2x+4)\ dx$

(f) $\displaystyle\int_0^1 (-6x+12)\ dx$

6. 

(a) $\displaystyle\int_0^2 f(x)\ dx$

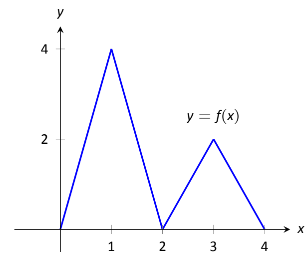
(b) $\displaystyle\int_0^3 f(x)\ dx$

(c) $\displaystyle\int_0^5 f(x)\ dx$

(d) $\displaystyle\int_2^5 f(x)\ dx$

(e) $\displaystyle\int_5^3 f(x)\ dx$

(f) $\displaystyle\int_0^3 -2f(x)\ dx$

7. 

(a) $\displaystyle\int_0^2 f(x)\ dx$

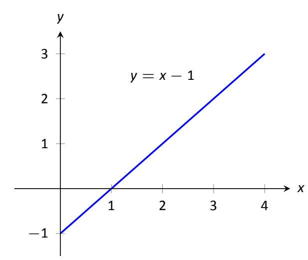
(b) $\displaystyle\int_2^4 f(x)\ dx$

(c) $\displaystyle\int_2^4 2f(x)\ dx$

(d) $\displaystyle\int_0^1 4x\ dx$

(e) $\displaystyle\int_2^3 (2x-4)\ dx$

(f) $\displaystyle\int_2^3 (4x-8)\ dx$

8. 

(a) $\displaystyle\int_0^1 (x-1)\ dx$

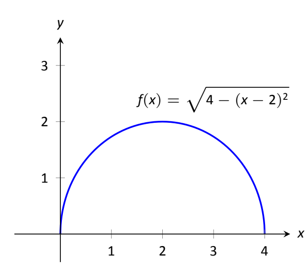
(b) $\displaystyle\int_0^2 (x-1)\ dx$

(c) $\displaystyle\int_0^3 (x-1)\ dx$

(d) $\displaystyle\int_2^3 (x-1)\ dx$

(e) $\displaystyle\int_1^4 (x-1)\ dx$

(f) $\displaystyle\int_1^4 \big((x-1)+1\big)\ dx$

9. 

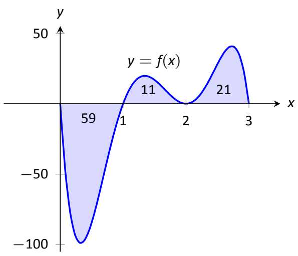
(a) $\displaystyle\int_0^2 f(x)\ dx$

(b) $\displaystyle\int_2^4 f(x)\ dx$

(c) $\displaystyle\int_0^4 f(x)\ dx$

(d) $\displaystyle\int_0^4 5f(x)\ dx$

In Exercises 10 – 13, a graph of a function $f(x)$ is given; the numbers inside the shaded regions give the area of that region. Evaluate the definite integrals using this area information.

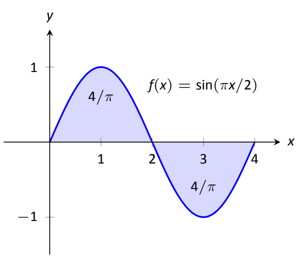
10. 

(a) $\int_0^1 f(x)\ dx$

(b) $\int_0^2 f(x)\ dx$

(c) $\int_0^3 f(x)\ dx$

(d) $\int_1^2 -3f(x)\ dx$

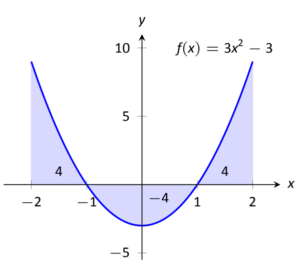
11. 

(a) $\int_0^2 f(x)\ dx$

(b) $\int_2^4 f(x)\ dx$

(c) $\int_0^4 f(x)\ dx$

(d) $\int_0^1 f(x)\ dx$

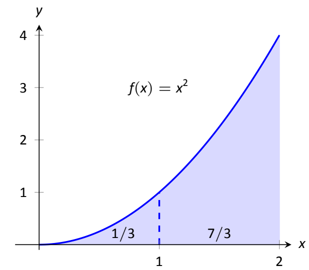
12. 

(a) $\int_{-2}^{-1} f(x)\ dx$

(b) $\int_1^2 f(x)\ dx$

(c) $\int_{-1}^1 f(x)\ dx$

(d) $\int_0^1 f(x)\ dx$

13. 

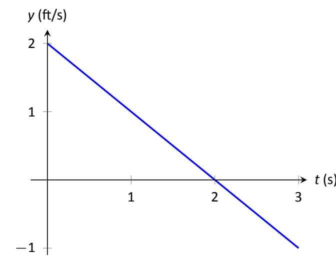
(a) $\int_0^2 5x^2\ dx$

(b) $\int_0^2 (x^2+3)\ dx$

(c) $\int_1^3 (x-1)^2\ dx$

(d) $\int_2^4 \left((x-2)^2+5\right)\ dx$

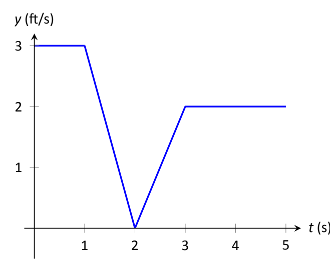
In Exercises 14 – 15, a graph of the velocity function of an object moving in a straight line is given. Answer the questions based on that graph.

14. 

(a) What is the object's maximum velocity?

(b) What is the object's maximum displacement?

(c) What is the object's total displacement on $[0,3]$?

15. 

(a) What is the object's maximum velocity?

(b) What is the object's maximum displacement?

(c) What is the object's total displacement on $[0,5]$?

16. An object is thrown straight up with a velocity, in ft/s, given by $v(t) = -32t + 64$, where t is in seconds, from a height of 48 feet.

(a) What is the object's maximum velocity?

(b) What is the object's maximum displacement?

(c) When does the maximum displacement occur?

(d) When will the object reach a height of 0? (Hint: find when the displacement is -48ft.)

17. An object is thrown straight up with a velocity, in ft/s, given by $v(t) = -32t + 96$, where t is in seconds, from a height of 64 feet.

    (a) What is the object's initial velocity?

    (b) When is the object's displacement 0?

    (c) How long does it take for the object to return to its initial height?

    (d) When will the object reach a height of 210 feet?

In Exercises 18 – 21, let

- $\displaystyle\int_0^2 f(x)\ dx = 5,$
- $\displaystyle\int_0^3 f(x)\ dx = 7,$
- $\displaystyle\int_0^2 g(x)\ dx = -3,$ **and**
- $\displaystyle\int_2^3 g(x)\ dx = 5.$

Use these values to evaluate the given definite integrals.

18. $\displaystyle\int_0^2 \big(f(x) + g(x)\big)\ dx$

19. $\displaystyle\int_0^3 \big(f(x) - g(x)\big)\ dx$

20. $\displaystyle\int_2^3 \big(3f(x) + 2g(x)\big)\ dx$

21. Find values for a and b such that
    $\displaystyle\int_0^3 \big(af(x) + bg(x)\big)\ dx = 0$

In Exercises 22 – 25, let

- $\displaystyle\int_0^3 s(t)\ dt = 10,$
- $\displaystyle\int_3^5 s(t)\ dt = 8,$
- $\displaystyle\int_3^5 r(t)\ dt = -1,$ **and**
- $\displaystyle\int_0^5 r(t)\ dt = 11.$

Use these values to evaluate the given definite integrals.

22. $\displaystyle\int_0^3 \big(s(t) + r(t)\big)\ dt$

23. $\displaystyle\int_5^0 \big(s(t) - r(t)\big)\ dt$

24. $\displaystyle\int_3^3 \big(\pi s(t) - 7r(t)\big)\ dt$

25. Find values for a and b such that
    $\displaystyle\int_0^5 \big(ar(t) + bs(t)\big)\ dt = 0$

Review

In Exercises 26 – 29, evaluate the given indefinite integral.

26. $\displaystyle\int \left(x^3 - 2x^2 + 7x - 9\right)\ dx$

27. $\displaystyle\int \left(\sin x - \cos x + \sec^2 x\right)\ dx$

28. $\displaystyle\int \left(\sqrt[3]{t} + \frac{1}{t^2} + 2^t\right)\ dt$

29. $\displaystyle\int \left(\frac{1}{x} - \csc x \cot x\right)\ dx$

1.3 Riemann Sums

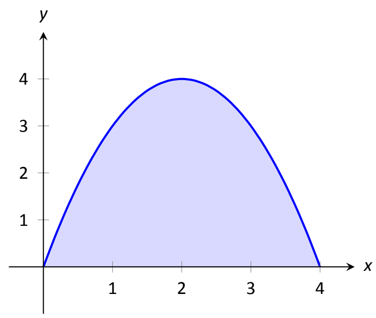


Figure 1.11: A graph of $f(x) = 4x - x^2$. What is the area of the shaded region?

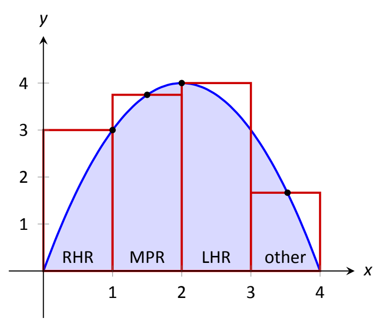


Figure 1.12: Approximating $\int_0^4 (4x - x^2)\ dx$ using rectangles. The heights of the rectangles are determined using different rules.

In the previous section we defined the definite integral of a function on $[a, b]$ to be the signed area between the curve and the x–axis. Some areas were simple to compute; we ended the section with a region whose area was not simple to compute. In this section we develop a technique to find such areas.

A fundamental calculus technique is to first answer a given problem with an approximation, then refine that approximation to make it better, then use limits in the refining process to find the exact answer. That is exactly what we will do here.

Consider the region given in Figure 1.11, which is the area under $y = 4x - x^2$ on $[0, 4]$. What is the signed area of this region – i.e., what is $\int_0^4 (4x - x^2)\ dx$?

We start by approximating. We can surround the region with a rectangle with height and width of 4 and find the area is approximately 16 square units. This is obviously an *over–approximation*; we are including area in the rectangle that is not under the parabola.

We have an approximation of the area, using one rectangle. How can we refine our approximation to make it better? The key to this section is this answer: *use more rectangles.*

Let's use 4 rectangles of equal width of 1. This *partitions* the interval $[0, 4]$ into 4 *subintervals*, $[0, 1]$, $[1, 2]$, $[2, 3]$ and $[3, 4]$. On each subinterval we will draw a rectangle.

There are three common ways to determine the height of these rectangles: the **Left Hand Rule**, the **Right Hand Rule**, and the **Midpoint Rule**. The **Left Hand Rule** says to evaluate the function at the left–hand endpoint of the subinterval and make the rectangle that height. In Figure 1.12, the rectangle drawn on the interval $[2, 3]$ has height determined by the Left Hand Rule; it has a height of $f(2)$. (The rectangle is labeled "LHR.")

The **Right Hand Rule** says the opposite: on each subinterval, evaluate the function at the right endpoint and make the rectangle that height. In the figure, the rectangle drawn on $[0, 1]$ is drawn using $f(1)$ as its height; this rectangle is labeled "RHR.".

The **Midpoint Rule** says that on each subinterval, evaluate the function at the midpoint and make the rectangle that height. The rectangle drawn on $[1, 2]$ was made using the Midpoint Rule, with a height of $f(1.5)$. That rectangle is labeled "MPR."

These are the three most common rules for determining the heights of approximating rectangles, but one is not forced to use one of these three methods. The rectangle on $[3, 4]$ has a height of approximately $f(3.53)$,

Notes:

very close to the Midpoint Rule. It was chosen so that the area of the rectangle is *exactly* the area of the region under f on $[3,4]$. (Later you'll be able to figure how to do this, too.)

The following example will approximate the value of $\int_0^4 (4x - x^2)\ dx$ using these rules.

Example 1.3.1 Using the Left Hand, Right Hand and Midpoint Rules

Approximate the value of $\int_0^4 (4x - x^2)\ dx$ using the Left Hand Rule, the Right Hand Rule, and the Midpoint Rule, using 4 equally spaced subintervals.

Solution We break the interval $[0,4]$ into four subintervals as before. In Figure 1.13 we see 4 rectangles drawn on $f(x) = 4x - x^2$ using the Left Hand Rule. (The areas of the rectangles are given in each figure.) Note how in the first subinterval, $[0,1]$, the rectangle has height $f(0) = 0$. We add up the areas of each rectangle (height$\times$ width) for our Left Hand Rule approximation:

$$f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 = \\ 0 + 3 + 4 + 3 = 10.$$

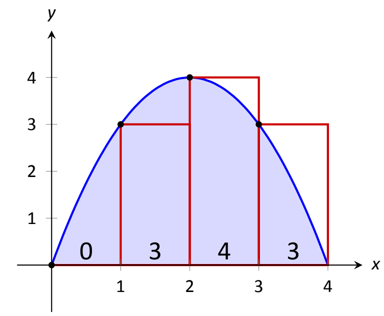


Figure 1.13: Approximating $\int_0^4 (4x - x^2)\ dx$ using the Left Hand Rule in Example 1.3.1.

Figure 1.14 shows 4 rectangles drawn under f using the Right Hand Rule; note how the $[3,4]$ subinterval has a rectangle of height 0.

In this example, these rectangle seem to be the mirror image of those found in Figure 1.13. (This is because of the symmetry of our shaded region.) Our approximation gives the same answer as before, though calculated a different way:

$$f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1 = \\ 3 + 4 + 3 + 0 = 10.$$

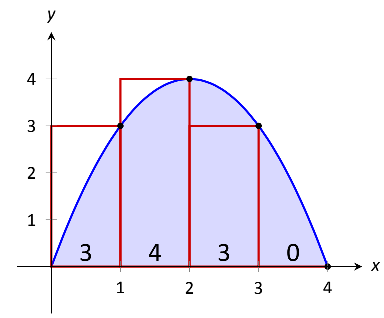


Figure 1.14: Approximating $\int_0^4 (4x - x^2)\ dx$ using the Right Hand Rule in Example 1.3.1.

Figure 1.15 shows 4 rectangles drawn under f using the Midpoint Rule. This gives an approximation of $\int_0^4 (4x - x^2)\ dx$ as:

$$f(0.5)\cdot 1 + f(1.5)\cdot 1 + f(2.5)\cdot 1 + f(3.5)\cdot 1 = \\ 1.75 + 3.75 + 3.75 + 1.75 = 11.$$

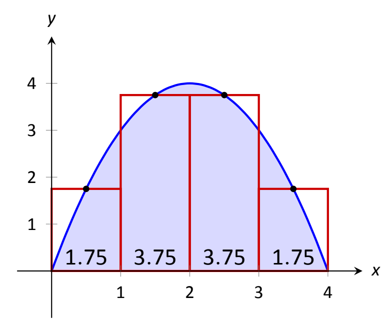


Figure 1.15: Approximating $\int_0^4 (4x - x^2)\ dx$ using the Midpoint Rule in Example 1.3.1.

Our three methods provide two approximations of $\int_0^4 (4x - x^2)\ dx$: 10 and 11.

Summation Notation

Notes:

Suppose we wish to add up a list of numbers $a_1, a_2, a_3, \ldots, a_9$. Instead of writing

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9,$$

we use summation notation and write

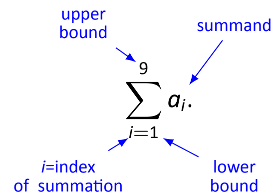


Figure 1.16: Understanding summation notation.

The upper case sigma represents the term "sum." The index of summation in this example is i; any symbol can be used. By convention, the index takes on only the integer values between (and including) the lower and upper bounds.

Let's practice using this notation.

Example 1.3.2 Using summation notation

Let the numbers $\{a_i\}$ be defined as $a_i = 2i - 1$ for integers i, where $i \geq 1$. So $a_1 = 1$, $a_2 = 3$, $a_3 = 5$, etc. (The output is the positive odd integers). Evaluate the following summations:

1. $\displaystyle\sum_{i=1}^{6} a_i$
2. $\displaystyle\sum_{i=3}^{7} (3a_i - 4)$
3. $\displaystyle\sum_{i=1}^{4} (a_i)^2$

Solution

1.
$$\begin{aligned}\sum_{i=1}^{6} a_i &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \\ &= 1 + 3 + 5 + 7 + 9 + 11 \\ &= 36.\end{aligned}$$

2. Note the starting value is different than 1:

$$\begin{aligned}\sum_{i=3}^{7} a_i &= (3a_3 - 4) + (3a_4 - 4) + (3a_5 - 4) + (3a_6 - 4) + (3a_7 - 4) \\ &= 11 + 17 + 23 + 29 + 35 \\ &= 115.\end{aligned}$$

Notes:

3.

$$\sum_{i=1}^{4}(a_i)^2 = (a_1)^2 + (a_2)^2 + (a_3)^2 + (a_4)^2$$
$$= 1^2 + 3^2 + 5^2 + 7^2$$
$$= 84$$

It might seem odd to stress a new, concise way of writing summations only to write each term out as we add them up. It is. The following theorem gives some of the properties of summations that allow us to work with them without writing individual terms. Examples will follow.

Theorem 1.3.1 Properties of Summations

1. $\displaystyle\sum_{i=1}^{n} c = c\cdot n$, where c is a constant.
2. $\displaystyle\sum_{i=m}^{n}(a_i \pm b_i) = \sum_{i=m}^{n} a_i \pm \sum_{i=m}^{n} b_i$
3. $\displaystyle\sum_{i=m}^{n} c\cdot a_i = c\cdot\sum_{i=m}^{n} a_i$
4. $\displaystyle\sum_{i=m}^{j} a_i + \sum_{i=j+1}^{n} a_i = \sum_{i=m}^{n} a_i$
5. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
6. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
7. $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

Example 1.3.3 Evaluating summations using Theorem 1.3.1
Revisit Example 1.3.2 and, using Theorem 1.3.1, evaluate

$$\sum_{i=1}^{6} a_i = \sum_{i=1}^{6}(2i-1).$$

Notes:

Solution

$$\begin{aligned}\sum_{i=1}^{6}(2i-1) &= \sum_{i=1}^{6}2i - \sum_{i=1}^{6}(1)\\ &= \left(2\sum_{i=1}^{6}i\right) - 6\\ &= 2\frac{6(6+1)}{2} - 6\\ &= 42-6=36\end{aligned}$$

We obtained the same answer without writing out all six terms. When dealing with small sizes of n, it may be faster to write the terms out by hand. However, Theorem 1.3.1 is incredibly important when dealing with large sums as we'll soon see.

Riemann Sums

Consider again $\int_0^4 (4x - x^2)\ dx$. We will approximate this definite integral using 16 equally spaced subintervals and the Right Hand Rule in Example 1.3.4. Before doing so, it will pay to do some careful preparation.

Figure 1.17 shows a number line of $[0, 4]$ divided into 16 equally spaced subintervals. We denote 0 as x_1; we have marked the values of x_5, x_9, x_{13} and x_{17}. We could mark them all, but the figure would get crowded. While it is easy to figure that $x_{10} = 2.25$, in general, we want a method of determining the value of x_i without consulting the figure. Consider:

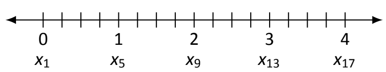
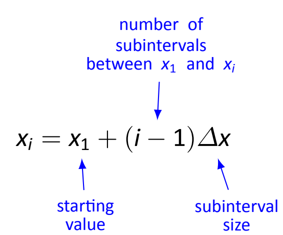


Figure 1.17: Dividing $[0, 4]$ into 16 equally spaced subintervals.



So $x_{10} = x_1 + 9(4/16) = 2.25$.

If we had partitioned $[0, 4]$ into 100 equally spaced subintervals, each subinterval would have length $\Delta x = 4/100 = 0.04$. We could compute x_{32} as

$$x_{32} = x_1 + 31(4/100) = 1.24.$$

(That was far faster than creating a sketch first.)

Notes:

Given any subdivision of $[0, 4]$, the first subinterval is $[x_1, x_2]$; the second is $[x_2, x_3]$; the i th subinterval is $[x_i, x_{i+1}]$.

When using the Left Hand Rule, the height of the i th rectangle will be $f(x_i)$.

When using the Right Hand Rule, the height of the i th rectangle will be $f(x_{i+1})$.

When using the Midpoint Rule, the height of the i th rectangle will be $f\left(\dfrac{x_i + x_{i+1}}{2}\right)$.

Thus approximating $\int_0^4 (4x - x^2)\, dx$ with 16 equally spaced subintervals can be expressed as follows, where $\Delta x = 4/16 = 1/4$:

Left Hand Rule: $\displaystyle\sum_{i=1}^{16} f(x_i)\Delta x$

Right Hand Rule: $\displaystyle\sum_{i=1}^{16} f(x_{i+1})\Delta x$

Midpoint Rule: $\displaystyle\sum_{i=1}^{16} f\left(\frac{x_i + x_{i+1}}{2}\right)\Delta x$

We use these formulas in the next two examples. The following example lets us practice using the Right Hand Rule and the summation formulas introduced in Theorem 1.3.1.

Example 1.3.4 Approximating definite integrals using sums

Approximate $\int_0^4 (4x - x^2)\, dx$ using the Right Hand Rule and summation formulas with 16 and 1000 equally spaced intervals.

Solution Using the formula derived before, using 16 equally spaced intervals and the Right Hand Rule, we can approximate the definite integral as

$$\sum_{i=1}^{16} f(x_{i+1})\Delta x.$$

We have $\Delta x = 4/16 = 0.25$. Since $x_i = 0 + (i - 1)\Delta x$, we have

$$\begin{aligned} x_{i+1} &= 0 + \big((i + 1) - 1\big)\Delta x \\ &= i\Delta x \end{aligned}$$

Notes:

Using the summation formulas, consider:

$$\begin{aligned}\int_0^4 (4x-x^2)\ dx &\approx \sum_{i=1}^{16} f(x_{i+1})\Delta x\\ &= \sum_{i=1}^{16} f(i\Delta x)\Delta x\\ &= \sum_{i=1}^{16} \big(4i\Delta x-(i\Delta x)^2\big)\Delta x\\ &= \sum_{i=1}^{16}(4i\Delta x^2-i^2\Delta x^3)\\ &= (4\Delta x^2)\sum_{i=1}^{16} i-\Delta x^3\sum_{i=1}^{16} i^2 \qquad (1.3)\\ &= (4\Delta x^2)\frac{16\cdot 17}{2}-\Delta x^3\frac{16(17)(33)}{6}\\ &= 4\cdot 0.25^2\cdot 136-0.25^3\cdot 1496\\ &= 10.625\end{aligned}$$

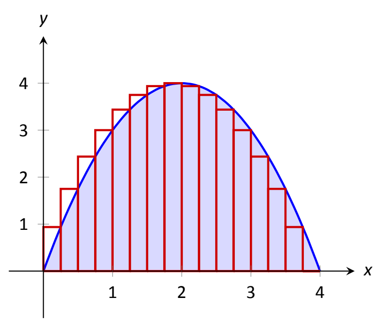


Figure 1.18: Approximating $\int_0^4 (4x-x^2)\ dx$ with the Right Hand Rule and 16 evenly spaced subintervals.

We were able to sum up the areas of 16 rectangles with very little computation. In Figure 1.18 the function and the 16 rectangles are graphed. While some rectangles over–approximate the area, other under–approximate the area (by about the same amount). Thus our approximate area of 10.625 is likely a fairly good approximation.

Notice Equation (1.3); by changing the 16's to 1,000's (and appropriately changing the value of Δx), we can use that equation to sum up 1000 rectangles! We do so here, skipping from the original summand to the equivalent of Equation (1.3) to save space. Note that $\Delta x = 4/1000 = 0.004$.

$$\begin{aligned}\int_0^4 (4x-x^2)\ dx &\approx \sum_{i=1}^{1000} f(x_{i+1})\Delta x\\ &= (4\Delta x^2)\sum_{i=1}^{1000} i-\Delta x^3\sum_{i=1}^{1000} i^2\\ &= (4\Delta x^2)\frac{1000\cdot 1001}{2}-\Delta x^3\frac{1000(1001)(2001)}{6}\\ &= 4\cdot 0.004^2\cdot 500500-0.004^3\cdot 333,833,500\\ &= 10.666656\end{aligned}$$

Notes:

Using many, many rectangles, we have a likely good approximation of $\int_0^4 (4x - x^2)\Delta x$. That is,

$$\int_0^4 (4x - x^2)\ dx \approx 10.666656.$$

Before the above example, we stated what the summations for the Left Hand, Right Hand and Midpoint Rules looked like. Each had the same basic structure, which was:

1. each rectangle has the same width, which we referred to as Δx, and
2. each rectangle's height is determined by evaluating f at a particular point in each subinterval. For instance, the Left Hand Rule states that each rectangle's height is determined by evaluating f at the left hand endpoint of the subinterval the rectangle lives on.

One could partition an interval $[a, b]$ with subintervals that did not have the same size. We refer to the length of the first subinterval as Δx_1, the length of the second subinterval as Δx_2, and so on, giving the length of the i^{th} subinterval as Δx_i. Also, one could determine each rectangle's height by evaluating f at *any* point in the i^{th} subinterval. We refer to the point picked in the first subinterval as c_1, the point picked in the second subinterval as c_2, and so on, with c_i representing the point picked in the i^{th} subinterval. Thus the height of the i^{th} subinterval would be $f(c_i)$, and the area of the i^{th} rectangle would be $f(c_i)\Delta x_i$.

Summations of rectangles with area $f(c_i)\Delta x_i$ are named after mathematician Georg Friedrich Bernhard Riemann, as given in the following definition.

Definition 1.3.1 Riemann Sum

Let f be defined on the closed interval $[a, b]$ and let Δx be a partition of $[a, b]$, with

$$a = x_1 < x_2 < \ldots < x_n < x_{n+1} = b.$$

Let Δx_i denote the length of the i^{th} subinterval $[x_i, x_{i+1}]$ and let c_i denote any value in the i^{th} subinterval.
The sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i$$

is a **Riemann sum** of f on $[a, b]$.

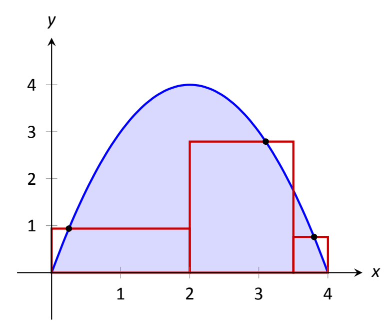


Figure 1.19: An example of a general Riemann sum to approximate $\int_0^4 (4x - x^2)\ dx$.

Notes:

Figure 1.19 shows the approximating rectangles of a Riemann sum of $\int_0^4 (4x - x^2)\ dx$. While the rectangles in this example do not approximate well the shaded area, they demonstrate that the subinterval widths may vary and the heights of the rectangles can be determined without following a particular rule.

"Usually" Riemann sums are calculated using one of the three methods we have introduced. The uniformity of construction makes computations easier. Before working another example, let's summarize some of what we have learned in a convenient way.

Key Idea 1.3.1 Riemann Sum Concepts

Consider $\displaystyle\int_a^b f(x)\ dx \approx \sum_{i=1}^{n} f(c_i)\Delta x_i$.

1. When the n subintervals have equal length, $\Delta x_i = \Delta x = \dfrac{b-a}{n}$.
2. The i th term of the partition is $x_i = a + (i-1)\Delta x$. (This makes $x_{n+1} = b$.)
3. The Left Hand Rule summation is: $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x$.
4. The Right Hand Rule summation is: $\displaystyle\sum_{i=1}^{n} f(x_{i+1})\Delta x$.
5. The Midpoint Rule summation is: $\displaystyle\sum_{i=1}^{n} f\left(\frac{x_i + x_{x+1}}{2}\right)\Delta x$.

Let's do another example.

Example 1.3.5 Approximating definite integrals with sums

Approximate $\int_{-2}^{3} (5x+2)\ dx$ using the Midpoint Rule and 10 equally spaced intervals.

Solution Following Key Idea 1.3.1, we have

$$\Delta x = \frac{3-(-2)}{10} = 1/2 \quad \text{and} \quad x_i = (-2) + (1/2)(i-1) = i/2 - 5/2.$$

Notes:

As we are using the Midpoint Rule, we will also need x_{i+1} and $\dfrac{x_i + x_{i+1}}{2}$. Since $x_i = i/2 - 5/2, \quad x_{i+1} = (i+1)/2 - 5/2 = i/2 - 2$. This gives

$$\frac{x_i + x_{i+1}}{2} = \frac{(i/2 - 5/2) + (i/2 - 2)}{2} = \frac{i - 9/2}{2} = i/2 - 9/4.$$

We now construct the Riemann sum and compute its value using summation formulas.

$$\begin{aligned}
\int_{-2}^{3} (5x+2)\ dx &\approx \sum_{i=1}^{10} f\left(\frac{x_i + x_{i+1}}{2}\right)\Delta x \\
&= \sum_{i=1}^{10} f(i/2 - 9/4)\Delta x \\
&= \sum_{i=1}^{10} \big(5(i/2 - 9/4) + 2\big)\Delta x \\
&= \Delta x \sum_{i=1}^{10} \left[\left(\frac{5}{2}\right) i - \frac{37}{4}\right] \\
&= \Delta x \left(\frac{5}{2}\sum_{i=1}^{10}(i) - \sum_{i=1}^{10}\left(\frac{37}{4}\right)\right) \\
&= \frac{1}{2}\left(\frac{5}{2}\cdot\frac{10(11)}{2} - 10\cdot\frac{37}{4}\right) \\
&= \frac{45}{2} = 22.5
\end{aligned}$$

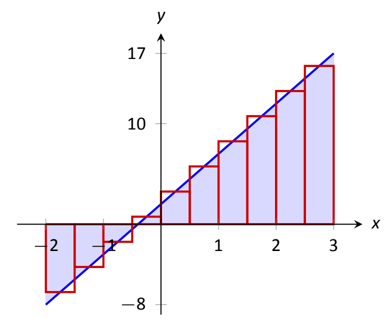


Figure 1.20: Approximating $\int_{-2}^{3}(5x+2)\ dx$ using the Midpoint Rule and 10 evenly spaced subintervals in Example 1.3.5.

Note the graph of $f(x) = 5x + 2$ in Figure 1.20. The regions whose area is computed by the definite integral are triangles, meaning we can find the exact answer without summation techniques. We find that the exact answer is indeed 22.5. One of the strengths of the Midpoint Rule is that often each rectangle includes area that should not be counted, but misses other area that should. When the partition size is small, these two amounts are about equal and these errors almost "cancel each other out." In this example, since our function is a line, these errors are exactly equal and they do cancel each other out, giving us the exact answer.

Note too that when the function is negative, the rectangles have a "negative" height. When we compute the area of the rectangle, we use $f(c_i)\Delta x$; when f is negative, the area is counted as negative.

Notice in the previous example that while we used 10 equally spaced intervals, the number "10" didn't play a big role in the calculations until

Notes:

the very end. Mathematicians love to abstract ideas; let's approximate the area of another region using n subintervals, where we do not specify a value of n until the very end.

Example 1.3.6 Approximating definite integrals with a formula, using sums

Revisit $\int_0^4 (4x - x^2)\,dx$ yet again. Approximate this definite integral using the Right Hand Rule with n equally spaced subintervals.

Solution Using Key Idea 1.3.1, we know $\Delta x = \frac{4-0}{n} = 4/n$. We also find $x_i = 0 + \Delta x(i-1) = 4(i-1)/n$. The Right Hand Rule uses x_{i+1}, which is $x_{i+1} = 4i/n$.

We construct the Right Hand Rule Riemann sum as follows. Be sure to follow each step carefully. If you get stuck, and do not understand how one line proceeds to the next, you may skip to the result and consider how this result is used. You should come back, though, and work through each step for full understanding.

$$
\begin{aligned}
\int_0^4 (4x - x^2)\,dx &\approx \sum_{i=1}^{n} f(x_{i+1})\Delta x \\
&= \sum_{i=1}^{n} f\left(\frac{4i}{n}\right)\Delta x \\
&= \sum_{i=1}^{n} \left[4\frac{4i}{n} - \left(\frac{4i}{n}\right)^2\right]\Delta x \\
&= \sum_{i=1}^{n} \left(\frac{16\Delta x}{n}\right) i - \sum_{i=1}^{n} \left(\frac{16\Delta x}{n^2}\right) i^2 \\
&= \left(\frac{16\Delta x}{n}\right)\sum_{i=1}^{n} i - \left(\frac{16\Delta x}{n^2}\right)\sum_{i=1}^{n} i^2 \\
&= \left(\frac{16\Delta x}{n}\right)\cdot\frac{n(n+1)}{2} - \left(\frac{16\Delta x}{n^2}\right)\frac{n(n+1)(2n+1)}{6} \qquad \left(\begin{matrix}\text{recall}\\ \Delta x = 4/n\end{matrix}\right) \\
&= \frac{32(n+1)}{n} - \frac{32(n+1)(2n+1)}{3n^2} \qquad \text{(now simplify)} \\
&= \frac{32}{3}\left(1 - \frac{1}{n^2}\right)
\end{aligned}
$$

The result is an amazing, easy to use formula. To approximate the definite integral with 10 equally spaced subintervals and the Right Hand

Notes:

Rule, set $n = 10$ and compute

$$\int_0^4 (4x - x^2)\ dx \approx \frac{32}{3}\left(1 - \frac{1}{10^2}\right) = 10.56.$$

Recall how earlier we approximated the definite integral with 4 subintervals; with $n = 4$, the formula gives 10, our answer as before.

It is now easy to approximate the integral with 1,000,000 subintervals! Hand-held calculators will round off the answer a bit prematurely giving an answer of 10.66666667. (The actual answer is 10.666666666656.)

We now take an important leap. Up to this point, our mathematics has been limited to geometry and algebra (finding areas and manipulating expressions). Now we apply *calculus*. For any *finite* n, we know that

$$\int_0^4 (4x - x^2)\ dx \approx \frac{32}{3}\left(1 - \frac{1}{n^2}\right).$$

Both common sense and high–level mathematics tell us that as n gets large, the approximation gets better. In fact, if we take the *limit* as $n \to \infty$, we get the *exact area* described by $\int_0^4 (4x - x^2)\ dx$. That is,

$$\begin{aligned}\int_0^4 (4x - x^2)\ dx &= \lim_{n\to\infty} \frac{32}{3}\left(1 - \frac{1}{n^2}\right)\\ &= \frac{32}{3}(1 - 0)\\ &= \frac{32}{3} = 10.\overline{6}\end{aligned}$$

This is a fantastic result. By considering n equally–spaced subintervals, we obtained a formula for an approximation of the definite integral that involved our variable n. As n grows large – without bound – the error shrinks to zero and we obtain the exact area.

This section started with a fundamental calculus technique: make an approximation, refine the approximation to make it better, then use limits in the refining process to get an exact answer. That is precisely what we just did.

Let's practice this again.

Example 1.3.7 Approximating definite integrals with a formula, using sums

Find a formula that approximates $\int_{-1}^5 x^3\ dx$ using the Right Hand Rule and n equally spaced subintervals, then take the limit as $n \to \infty$ to find the exact area.

Notes:

Solution Following Key Idea 1.3.1, we have $\Delta x = \frac{5-(-1)}{n} = 6/n$. We have $x_i = (-1) + (i-1)\Delta x$; as the Right Hand Rule uses x_{i+1}, we have $x_{i+1} = (-1) + i\Delta x$.

The Riemann sum corresponding to the Right Hand Rule is (followed by simplifications):

$$\begin{aligned}
\int_{-1}^{5} x^3\,dx &\approx \sum_{i=1}^{n} f(x_{i+1})\Delta x \\
&= \sum_{i=1}^{n} f(-1+i\Delta x)\Delta x \\
&= \sum_{i=1}^{n} (-1+i\Delta x)^3\Delta x \\
&= \sum_{i=1}^{n} \big((i\Delta x)^3 - 3(i\Delta x)^2 + 3i\Delta x - 1\big)\Delta x \quad \text{(now distribute } \Delta x\text{)} \\
&= \sum_{i=1}^{n} \big(i^3\Delta x^4 - 3i^2\Delta x^3 + 3i\Delta x^2 - \Delta x\big) \quad \text{(now split up summation)} \\
&= \Delta x^4 \sum_{i=1}^{n} i^3 - 3\Delta x^3 \sum_{i=1}^{n} i^2 + 3\Delta x^2 \sum_{i=1}^{n} i - \sum_{i=1}^{n} \Delta x \\
&= \Delta x^4 \left(\frac{n(n+1)}{2}\right)^2 - 3\Delta x^3 \frac{n(n+1)(2n+1)}{6} + 3\Delta x^2 \frac{n(n+1)}{2} - n\Delta x
\end{aligned}$$

(use $\Delta x = 6/n$)

$$= \frac{1296}{n^4}\cdot\frac{n^2(n+1)^2}{4} - 3\frac{216}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + 3\frac{36}{n^2}\frac{n(n+1)}{2} - 6$$

(now do a sizable amount of algebra to simplify)

$$= 156 + \frac{378}{n} + \frac{216}{n^2}$$

Once again, we have found a compact formula for approximating the definite integral with n equally spaced subintervals and the Right Hand Rule. Using 10 subintervals, we have an approximation of 195.96 (these rectangles are shown in Figure 1.21). Using $n = 100$ gives an approximation of 159.802.

Now find the exact answer using a limit:

$$\int_{-1}^{5} x^3\,dx = \lim_{n\to\infty}\left(156 + \frac{378}{n} + \frac{216}{n^2}\right) = 156.$$

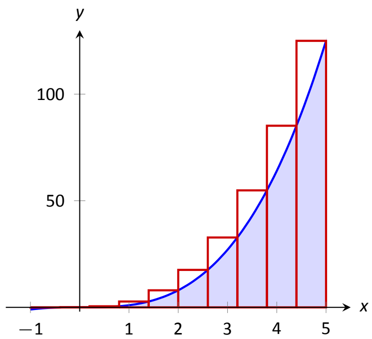


Figure 1.21: Approximating $\int_{-1}^{5} x^3\,dx$ using the Right Hand Rule and 10 evenly spaced subintervals.

Notes:

Limits of Riemann Sums

We have used limits to evaluate exactly given definite limits. Will this always work? We will show, given not–very–restrictive conditions, that yes, it will always work.

The previous two examples demonstrated how an expression such as

$$\sum_{i=1}^{n} f(x_{i+1})\Delta x$$

can be rewritten as an expression explicitly involving n, such as $32/3(1 - 1/n^2)$.

Viewed in this manner, we can think of the summation as a function of n. An n value is given (where n is a positive integer), and the sum of areas of n equally spaced rectangles is returned, using the Left Hand, Right Hand, or Midpoint Rules.

Given a definite integral $\int_a^b f(x)\, dx$, let:

- $S_L(n) = \displaystyle\sum_{i=1}^{n} f(x_i)\Delta x$, the sum of equally spaced rectangles formed using the Left Hand Rule,
- $S_R(n) = \displaystyle\sum_{i=1}^{n} f(x_{i+1})\Delta x$, the sum of equally spaced rectangles formed using the Right Hand Rule, and
- $S_M(n) = \displaystyle\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1}}{2}\right)\Delta x$, the sum of equally spaced rectangles formed using the Midpoint Rule.

Recall the definition of a limit as $n \to \infty$: $\lim_{n\to\infty} S_L(n) = K$ if, given any $\epsilon > 0$, there exists $N > 0$ such that

$$|S_L(n) - K| < \epsilon \quad \text{when} \quad n \geq N.$$

The following theorem states that we can use any of our three rules to find the exact value of a definite integral $\int_a^b f(x)\, dx$. It also goes two steps further. The theorem states that the height of each rectangle doesn't have to be determined following a specific rule, but could be $f(c_i)$, where c_i is any point in the i th subinterval, as discussed before Riemann Sums where defined in Definition 1.3.1.

Notes:

The theorem goes on to state that the rectangles do not need to be of the same width. Using the notation of Definition 1.3.1, let Δx_i denote the length of the i th subinterval in a partition of $[a,b]$. Now let $||\Delta x||$ represent the length of the largest subinterval in the partition: that is, $||\Delta x||$ is the largest of all the Δx_i's. If $||\Delta x||$ is small, then $[a,b]$ must be partitioned into many subintervals, since all subintervals must have small lengths. "Taking the limit as $||\Delta x||$ goes to zero" implies that the number n of subintervals in the partition is growing to infinity, as the largest subinterval length is becoming arbitrarily small. We then interpret the expression

$$\lim_{||\Delta x|| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

as "the limit of the sum of rectangles, where the width of each rectangle can be different but getting small, and the height of each rectangle is not necessarily determined by a particular rule." The theorem states that this Riemann Sum also gives the value of the definite integral of f over $[a,b]$.

Theorem 1.3.2 Definite Integrals and the Limit of Riemann Sums

Let f be continuous on the closed interval $[a,b]$ and let $S_L(n)$, $S_R(n)$ and $S_M(n)$ be defined as before. Then:

1. $\lim_{n\to\infty} S_L(n) = \lim_{n\to\infty} S_R(n) = \lim_{n\to\infty} S_M(n) = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$,
2. $\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)\ dx$, and
3. $\lim_{\|\Delta x\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)\ dx$.

We summarize what we have learned over the past few sections here.

- Knowing the "area under the curve" can be useful. One common example is: the area under a velocity curve is displacement.
- We have defined the definite integral, $\int_a^b f(x)\ dx$, to be the signed area under f on the interval $[a,b]$.

Notes:

- While we can approximate a definite integral many ways, we have focused on using rectangles whose heights can be determined using: the Left Hand Rule, the Right Hand Rule and the Midpoint Rule.
- Sums of rectangles of this type are called Riemann sums.
- The exact value of the definite integral can be computed using the limit of a Riemann sum. We generally use one of the above methods as it makes the algebra simpler.

We first learned of derivatives through limits then learned rules that made the process simpler. We know of a way to evaluate a definite integral using limits; in the next section we will see how the Fundamental Theorem of Calculus makes the process simpler. The key feature of this theorem is its connection between the indefinite integral and the definite integral.

Notes:

Exercises 1.3

Terms and Concepts

1. A fundamental calculus technique is to use ________ to refine approximations to get an exact answer.

2. What is the upper bound in the summation $\sum_{i=7}^{14}(48i - 201)$?

3. This section approximates definite integrals using what geometric shape?

4. T/F: A sum using the Right Hand Rule is an example of a Riemann Sum.

Problems

In Exercises 5 – 11, write out each term of the summation and compute the sum.

5. $\sum_{i=2}^{4} i^2$

6. $\sum_{i=-1}^{3} (4i - 2)$

7. $\sum_{i=-2}^{2} \sin(\pi i/2)$

8. $\sum_{i=1}^{5} \frac{1}{i}$

9. $\sum_{i=1}^{6} (-1)^i i$

10. $\sum_{i=1}^{4} \left(\frac{1}{i} - \frac{1}{i+1}\right)$

11. $\sum_{i=0}^{5} (-1)^i \cos(\pi i)$

In Exercises 12 – 15, write each sum in summation notation.

12. $3 + 6 + 9 + 12 + 15$

13. $-1 + 0 + 3 + 8 + 15 + 24 + 35 + 48 + 63$

14. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5}$

15. $1 - e + e^2 - e^3 + e^4$

In Exercises 16 – 22, evaluate the summation using Theorem 1.3.1.

16. $\sum_{i=1}^{25} i$

17. $\sum_{i=1}^{10} (3i^2 - 2i)$

18. $\sum_{i=1}^{15} (2i^3 - 10)$

19. $\sum_{i=1}^{10} (-4i^3 + 10i^2 - 7i + 11)$

20. $\sum_{i=1}^{10} (i^3 - 3i^2 + 2i + 7)$

21. $1 + 2 + 3 + \ldots + 99 + 100$

22. $1 + 4 + 9 + \ldots + 361 + 400$

Theorem 1.3.1 states

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{k} a_i + \sum_{i=k+1}^{n} a_i \text{ , so}$$

$$\sum_{i=k+1}^{n} a_i = \sum_{i=1}^{n} a_i - \sum_{i=1}^{k} a_i \text{ .}$$

Use this fact, along with other parts of Theorem 1.3.1, to evaluate the summations given in Exercises 23 – 26.

23. $\sum_{i=11}^{20} i$

24. $\sum_{i=16}^{25} i^3$

25. $\sum_{i=7}^{12} 4$

26. $\sum_{i=5}^{10} 4i^3$

In Exercises 27 – 32, a definite integral $\int_a^b f(x)\ dx$ is given.

(a) Graph $f(x)$ on $[a,b]$.

(b) Add to the sketch rectangles using the provided rule.

(c) Approximate $\int_a^b f(x)\ dx$ by summing the areas of the rectangles.

27. $\int_{-3}^{3} x^2\ dx$, with 6 rectangles using the Left Hand Rule.

28. $\int_0^2 (5-x^2)\ dx$, with 4 rectangles using the Midpoint Rule.

29. $\int_0^{\pi} \sin x\ dx$, with 6 rectangles using the Right Hand Rule.

30. $\int_0^3 2^x\ dx$, with 5 rectangles using the Left Hand Rule.

31. $\int_1^2 \ln x\ dx$, with 3 rectangles using the Midpoint Rule.

32. $\int_1^9 \frac{1}{x}\ dx$, with 4 rectangles using the Right Hand Rule.

In Exercises 33 – 38, a definite integral $\int_a^b f(x)\ dx$ is given. As demonstrated in Examples 1.3.6 and 1.3.7, do the following.

(a) Find a formula to approximate $\int_a^b f(x)\ dx$ using n subintervals and the provided rule.

(b) Evaluate the formula using $n = 10$, 100 and $1,000$.

(c) Find the limit of the formula, as $n \to \infty$, to find the exact value of $\int_a^b f(x)\ dx$.

33. $\int_0^1 x^3\ dx$, using the Right Hand Rule.

34. $\int_{-1}^1 3x^2\ dx$, using the Left Hand Rule.

35. $\int_{-1}^3 (3x-1)\ dx$, using the Midpoint Rule.

36. $\int_1^4 (2x^2-3)\ dx$, using the Left Hand Rule.

37. $\int_{-10}^{10} (5-x)\ dx$, using the Right Hand Rule.

38. $\int_0^1 (x^3-x^2)\ dx$, using the Right Hand Rule.

Review

In Exercises 39 – 44, find an antiderivative of the given function.

39. $f(x) = 5\sec^2 x$

40. $f(x) = \dfrac{7}{x}$

41. $g(t) = 4t^5 - 5t^3 + 8$

42. $g(t) = 5 \cdot 8^t$

43. $g(t) = \cos t + \sin t$

44. $f(x) = \dfrac{1}{\sqrt{x}}$

1.4 The Fundamental Theorem of Calculus

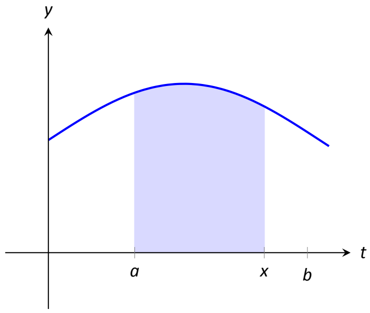


Figure 1.22: The area of the shaded region is $F(x) = \int_a^x f(t)\,dt$.

Let $f(t)$ be a continuous function defined on $[a, b]$. The definite integral $\int_a^b f(x)\,dx$ is the "area under f" on $[a, b]$. We can turn this concept into a function by letting the upper (or lower) bound vary.

Let $F(x) = \int_a^x f(t)\,dt$. It computes the area under f on $[a, x]$ as illustrated in Figure 1.22. We can study this function using our knowledge of the definite integral. For instance, $F(a) = 0$ since $\int_a^a f(t)\,dt = 0$.

We can also apply calculus ideas to $F(x)$; in particular, we can compute its derivative. While this may seem like an innocuous thing to do, it has far–reaching implications, as demonstrated by the fact that the result is given as an important theorem.

Theorem 1.4.1 The Fundamental Theorem of Calculus, Part 1

Let f be continuous on $[a, b]$ and let $F(x) = \int_a^x f(t)\,dt$. Then F is a differentiable function on (a, b), and

$$F'(x) = f(x).$$

Initially this seems simple, as demonstrated in the following example.

Example 1.4.1 Using the Fundamental Theorem of Calculus, Part 1

Let $F(x) = \displaystyle\int_{-5}^{x} (t^2 + \sin t)\,dt$. What is $F'(x)$?

Solution Using the Fundamental Theorem of Calculus, we have $F'(x) = x^2 + \sin x$.

This simple example reveals something incredible: $F(x)$ is an antiderivative of $x^2 + \sin x$! Therefore, $F(x) = \frac{1}{3}x^3 - \cos x + C$ for some value of C. (We can find C, but generally we do not care. We know that $F(-5) = 0$, which allows us to compute C. In this case, $C = \cos(-5) + \frac{125}{3}$.)

We have done more than found a complicated way of computing an antiderivative. Consider a function f defined on an open interval containing a, b and c. Suppose we want to compute $\int_a^b f(t)\,dt$. First, let $F(x) = \int_c^x f(t)\,dt$. Using the properties of the definite integral found in

Notes:

Theorem 1.2.1, we know

$$\begin{aligned}\int_a^b f(t)\ dt &= \int_a^c f(t)\ dt + \int_c^b f(t)\ dt \\ &= -\int_c^a f(t)\ dt + \int_c^b f(t)\ dt \\ &= -F(a) + F(b) \\ &= F(b) - F(a).\end{aligned}$$

We now see how indefinite integrals and definite integrals are related: we can evaluate a definite integral using antiderivatives! This is the second part of the Fundamental Theorem of Calculus.

Theorem 1.4.2 The Fundamental Theorem of Calculus, Part 2

Let f be continuous on $[a, b]$ and let F be *any* antiderivative of f. Then

$$\int_a^b f(x)\ dx = F(b) - F(a).$$

Example 1.4.2 Using the Fundamental Theorem of Calculus, Part 2

We spent a great deal of time in the previous section studying $\int_0^4 (4x - x^2)\ dx$. Using the Fundamental Theorem of Calculus, evaluate this definite integral.

Solution We need an antiderivative of $f(x) = 4x - x^2$. All antiderivatives of f have the form $F(x) = 2x^2 - \frac{1}{3}x^3 + C$; for simplicity, choose $C = 0$.

The Fundamental Theorem of Calculus states

$$\int_0^4 (4x-x^2)\ dx = F(4)-F(0) = \big(2(4)^2 - \frac{1}{3}4^3\big) - \big(0-0\big) = 32 - \frac{64}{3} = 32/3.$$

This is the same answer we obtained using limits in the previous section, just with much less work.

Notation: A special notation is often used in the process of evaluating definite integrals using the Fundamental Theorem of Calculus. Instead

Notes:

of explicitly writing $F(b) - F(a)$, the notation $F(x)\Big|_a^b$ is used. Thus the solution to Example 1.4.2 would be written as:

$$\int_0^4 (4x - x^2)\,dx = \left(2x^2 - \frac{1}{3}x^3\right)\Big|_0^4 = \big(2(4)^2 - \frac{1}{3}4^3\big) - \big(0 - 0\big) = 32/3.$$

The Constant C: *Any* antiderivative $F(x)$ can be chosen when using the Fundamental Theorem of Calculus to evaluate a definite integral, meaning any value of C can be picked. The constant *always* cancels out of the expression when evaluating $F(b) - F(a)$, so it does not matter what value is picked. This being the case, we might as well let $C = 0$.

Example 1.4.3 Using the Fundamental Theorem of Calculus, Part 2

Evaluate the following definite integrals.

1. $\displaystyle\int_{-2}^{2} x^3\,dx$ 2. $\displaystyle\int_0^{\pi} \sin x\,dx$ 3. $\displaystyle\int_0^5 e^t\,dt$ 4. $\displaystyle\int_4^9 \sqrt{u}\,du$ 5. $\displaystyle\int_1^5 2\,dx$

Solution

1. $\displaystyle\int_{-2}^{2} x^3\,dx = \frac{1}{4}x^4\Big|_{-2}^{2} = \left(\frac{1}{4}2^4\right) - \left(\frac{1}{4}(-2)^4\right) = 0.$

2. $\displaystyle\int_0^{\pi} \sin x\,dx = -\cos x\Big|_0^{\pi} = -\cos\pi - \big(-\cos 0\big) = 1 + 1 = 2.$

   (This is interesting; it says that the area under one "hump" of a sine curve is 2.)

3. $\displaystyle\int_0^5 e^t\,dt = e^t\Big|_0^5 = e^5 - e^0 = e^5 - 1 \approx 147.41.$

4. $\displaystyle\int_4^9 \sqrt{u}\,du = \int_4^9 u^{\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{3}{2}}\Big|_4^9 = \frac{2}{3}\left(9^{\frac{3}{2}} - 4^{\frac{3}{2}}\right) = \frac{2}{3}(27-8) = \frac{38}{3}.$

5. $\displaystyle\int_1^5 2\,dx = 2x\Big|_1^5 = 2(5) - 2 = 2(5-1) = 8.$

   This integral is interesting; the integrand is a constant function, hence we are finding the area of a rectangle with width $(5-1) = 4$ and height 2. Notice how the evaluation of the definite integral led to $2(4) = 8$.

   In general, if c is a constant, then $\int_a^b c\,dx = c(b-a)$.

Notes:

Understanding Motion with the Fundamental Theorem of Calculus

We established, starting with the fact that the derivative of a position function is a velocity function, and the derivative of a velocity function is an acceleration function. Now consider definite integrals of velocity and acceleration functions. Specifically, if $v(t)$ is a velocity function, what does $\int_a^b v(t)\ dt$ mean?

The Fundamental Theorem of Calculus states that

$$\int_a^b v(t)\ dt = V(b) - V(a),$$

where $V(t)$ is any antiderivative of $v(t)$. Since $v(t)$ is a velocity function, $V(t)$ must be a position function, and $V(b) - V(a)$ measures a change in position, or **displacement**.

Example 1.4.4 Finding displacement
A ball is thrown straight up with velocity given by $v(t) = -32t + 20\text{ft/s}$, where t is measured in seconds. Find, and interpret, $\int_0^1 v(t)\ dt$.

Solution Using the Fundamental Theorem of Calculus, we have

$$\begin{aligned} \int_0^1 v(t)\ dt &= \int_0^1 (-32t + 20)\ dt \\ &= -16t^2 + 20t\Big|_0^1 \\ &= 4. \end{aligned}$$

Thus if a ball is thrown straight up into the air with velocity $v(t) = -32t + 20$, the height of the ball, 1 second later, will be 4 feet above the initial height. (Note that the ball has *traveled* much farther. It has gone up to its peak and is falling down, but the difference between its height at $t = 0$ and $t = 1$ is 4ft.)

Integrating a rate of change function gives total change. Velocity is the rate of position change; integrating velocity gives the total change of position, i.e., displacement.

Notes:

Integrating a speed function gives a similar, though different, result. Speed is also the rate of position change, but does not account for direction. So integrating a speed function gives total change of position, without the possibility of "negative position change." Hence the integral of a speed function gives *distance traveled.*

As acceleration is the rate of velocity change, integrating an acceleration function gives total change in velocity. We do not have a simple term for this analogous to displacement. If $a(t) = 5\text{miles/h}^2$ and t is measured in hours, then

$$\int_0^3 a(t)\ dt = 15$$

means the velocity has increased by 15m/h from $t = 0$ to $t = 3$.

The Fundamental Theorem of Calculus and the Chain Rule

Part 1 of the Fundamental Theorem of Calculus (FTC) states that given $F(x) = \displaystyle\int_a^x f(t)\,dt$, $F'(x) = f(x)$. Using other notation, $\dfrac{d}{dx}\big(F(x)\big) = f(x)$. While we have just practiced evaluating definite integrals, sometimes finding antiderivatives is impossible and we need to rely on other techniques to approximate the value of a definite integral. Functions written as $F(x) = \int_a^x f(t)\ dt$ are useful in such situations.

It may be of further use to compose such a function with another. As an example, we may compose $F(x)$ with $g(x)$ to get

$$F\big(g(x)\big) = \int_a^{g(x)} f(t)\ dt.$$

What is the derivative of such a function? The Chain Rule can be employed to state

$$\frac{d}{dx}\Big(F\big(g(x)\big)\Big) = F'\big(g(x)\big)g'(x) = f\big(g(x)\big)g'(x).$$

An example will help us understand this.

Example 1.4.5 The FTC, Part 1, and the Chain Rule

Find the derivative of $F(x) = \displaystyle\int_2^{x^2} \ln t\ dt$.

Solution We can view $F(x)$ as being the function $G(x) = \displaystyle\int_2^x \ln t\,dt$ composed with $g(x) = x^2$; that is, $F(x) = G\big(g(x)\big)$. The Fundamental

Notes:

Theorem of Calculus states that $G'(x) = \ln x$. The Chain Rule gives us

$$\begin{aligned} F'(x) &= G'\big(g(x)\big)g'(x) \\ &= \ln(g(x))g'(x) \\ &= \ln(x^2)2x \\ &= 2x\ln x^2 \end{aligned}$$

Normally, the steps defining $G(x)$ and $g(x)$ are skipped.

Practice this once more.

Example 1.4.6 The FTC, Part 1, and the Chain Rule

Find the derivative of $F(x) = \displaystyle\int_{\cos x}^{5} t^3\ dt$.

Solution Note that $F(x) = -\displaystyle\int_{5}^{\cos x} t^3\ dt$. Viewed this way, the derivative of F is straightforward:

$$F'(x) = \sin x \cos^3 x.$$

Area Between Curves

Consider continuous functions $f(x)$ and $g(x)$ defined on $[a, b]$, where $f(x) \geq g(x)$ for all x in $[a, b]$, as demonstrated in Figure 1.23. What is the area of the shaded region bounded by the two curves over $[a, b]$?

The area can be found by recognizing that this area is "the area under f − the area under g." Using mathematical notation, the area is

$$\int_a^b f(x)\ dx - \int_a^b g(x)\ dx.$$

Properties of the definite integral allow us to simplify this expression to

$$\int_a^b \big(f(x) - g(x)\big)\ dx.$$

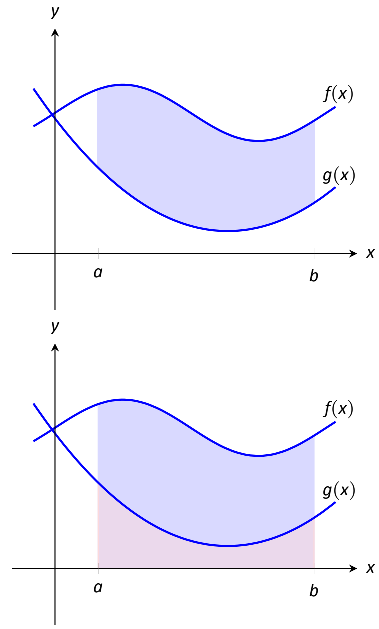


Figure 1.23: Finding the area bounded by two functions on an interval; it is found by subtracting the area under g from the area under f.

Notes:

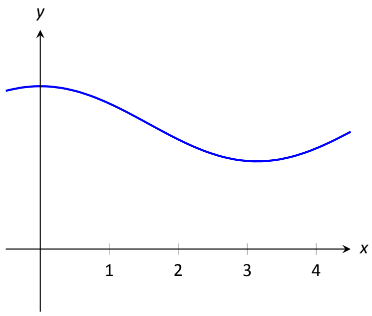


Figure 1.25: A graph of a function f to introduce the Mean Value Theorem.

Theorem 1.4.3 Area Between Curves

Let $f(x)$ and $g(x)$ be continuous functions defined on $[a,b]$ where $f(x) \geq g(x)$ for all x in $[a,b]$. The area of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is

$$\int_a^b \big(f(x) - g(x)\big)\ dx.$$

Example 1.4.7 Finding area between curves

Find the area of the region enclosed by $y = x^2 + x - 5$ and $y = 3x - 2$.

Solution It will help to sketch these two functions, as done in Figure 1.24. The region whose area we seek is completely bounded by these two functions; they seem to intersect at $x = -1$ and $x = 3$. To check, set $x^2 + x - 5 = 3x - 2$ and solve for x:

$$\begin{aligned} x^2 + x - 5 &= 3x - 2 \\ (x^2 + x - 5) - (3x - 2) &= 0 \\ x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0 \\ x &= -1,\ 3. \end{aligned}$$

Following Theorem 1.4.3, the area is

$$\begin{aligned} \int_{-1}^{3} \big(3x - 2 - (x^2 + x - 5)\big)\ dx &= \int_{-1}^{3} (-x^2 + 2x + 3)\ dx \\ &= \left(-\frac{1}{3}x^3 + x^2 + 3x\right)\Bigg|_{-1}^{3} \\ &= -\frac{1}{3}(27) + 9 + 9 - \left(\frac{1}{3} + 1 - 3\right) \\ &= 10\frac{2}{3} = 10.\overline{6} \end{aligned}$$

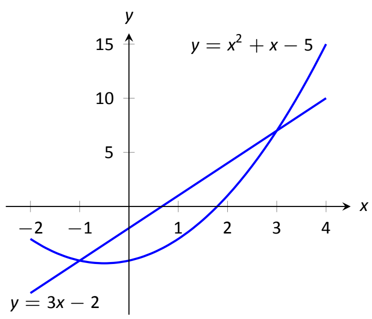


Figure 1.24: Sketching the region enclosed by $y = x^2 + x - 5$ and $y = 3x - 2$ in Example 1.4.7.

The Mean Value Theorem and Average Value

Consider the graph of a function f in Figure 1.25 and the area defined by $\int_1^4 f(x)\ dx$. Three rectangles are drawn in Figure 1.26; in (a), the

Notes:

height of the rectangle is greater than f on $[1,4]$, hence the area of this rectangle is is greater than $\int_0^4 f(x)\ dx$.

In (b), the height of the rectangle is smaller than f on $[1,4]$, hence the area of this rectangle is less than $\int_1^4 f(x)\ dx$.

Finally, in (c) the height of the rectangle is such that the area of the rectangle is *exactly* that of $\int_0^4 f(x)\ dx$. Since rectangles that are "too big", as in (a), and rectangles that are "too little," as in (b), give areas greater/lesser than $\int_1^4 f(x)\ dx$, it makes sense that there is a rectangle, whose top intersects $f(x)$ somewhere on $[1,4]$, whose area is *exactly* that of the definite integral.

We state this idea formally in a theorem.

Theorem 1.4.4 The Mean Value Theorem of Integration

Let f be continuous on $[a,b]$. There exists a value c in $[a,b]$ such that

$$\int_a^b f(x)\ dx = f(c)(b-a).$$

This is an *existential* statement; c exists, but we do not provide a method of finding it. Theorem 1.4.4 is directly connected to the Mean Value Theorem of Differentiation, given as Theorem **??**; we leave it to the reader to see how.

We demonstrate the principles involved in this version of the Mean Value Theorem in the following example.

Example 1.4.8 Using the Mean Value Theorem

Consider $\int_0^\pi \sin x\ dx$. Find a value c guaranteed by the Mean Value Theorem.

Solution We first need to evaluate $\int_0^\pi \sin x\ dx$. (This was previously done in Example 1.4.3.)

$$\int_0^\pi \sin x\ dx = -\cos x\Big|_0^\pi = 2.$$

Thus we seek a value c in $[0,\pi]$ such that $\pi \sin c = 2$.

$$\pi \sin c = 2 \quad \Rightarrow \quad \sin c = 2/\pi \quad \Rightarrow \quad c = \arcsin(2/\pi) \approx 0.69.$$

Notes:

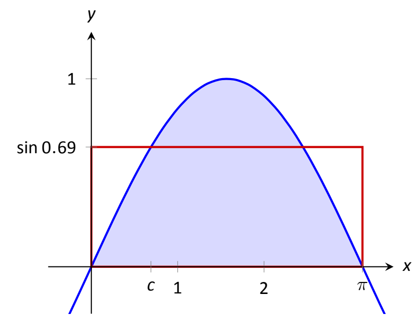
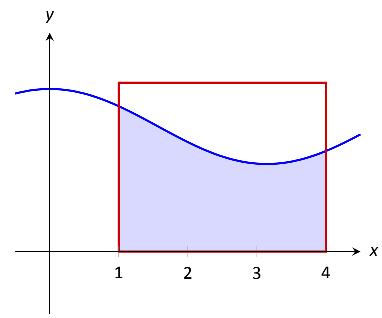
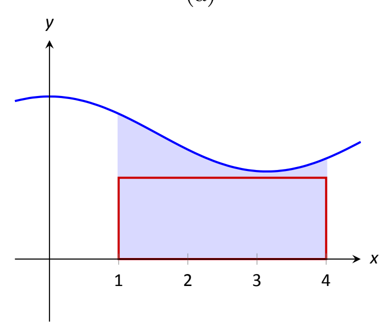


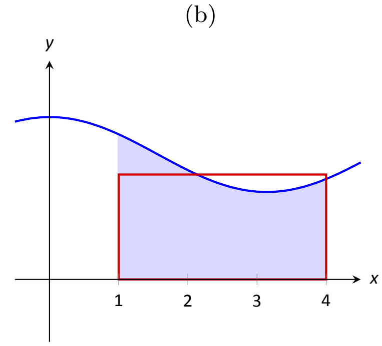
Figure 1.27: A graph of $y = \sin x$ on $[0,\pi]$ and the rectangle guaranteed by the Mean Value Theorem.



(a)



(b)



(c)

Figure 1.26: Differently sized rectangles give upper and lower bounds on $\int_1^4 f(x)\ dx$; the last rectangle matches the area exactly.

In Figure 1.27 $\sin x$ is sketched along with a rectangle with height $\sin(0.69)$. The area of the rectangle is the same as the area under $\sin x$ on $[0, \pi]$.

Let f be a function on $[a, b]$ with c such that $f(c)(b-a) = \int_a^b f(x)\ dx$. Consider $\int_a^b \big(f(x) - f(c)\big)\ dx$:

$$\begin{aligned}\int_a^b \big(f(x) - f(c)\big)\ dx &= \int_a^b f(x) - \int_a^b f(c)\ dx\\ &= f(c)(b-a) - f(c)(b-a)\\ &= 0.\end{aligned}$$

When $f(x)$ is shifted by $-f(c)$, the amount of area under f above the x–axis on $[a, b]$ is the same as the amount of area below the x–axis above f; see Figure 1.28 for an illustration of this. In this sense, we can say that $f(c)$ is the *average value* of f on $[a, b]$.

The value $f(c)$ is the average value in another sense. First, recognize that the Mean Value Theorem can be rewritten as

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\ dx,$$

for some value of c in $[a, b]$. Next, partition the interval $[a, b]$ into n equally spaced subintervals, $a = x_1 < x_2 < \ldots < x_{n+1} = b$ and choose any c_i in $[x_i, x_{i+1}]$. The average of the numbers $f(c_1)$, $f(c_2)$, …, $f(c_n)$ is:

$$\frac{1}{n}\Big(f(c_1) + f(c_2) + \ldots + f(c_n)\Big) = \frac{1}{n}\sum_{i=1}^{n} f(c_i).$$

Multiply this last expression by 1 in the form of $\frac{(b-a)}{(b-a)}$:

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} f(c_i) &= \sum_{i=1}^{n} f(c_i)\frac{1}{n}\\ &= \sum_{i=1}^{n} f(c_i)\frac{1}{n}\frac{(b-a)}{(b-a)}\\ &= \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\frac{b-a}{n}\\ &= \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\Delta x \quad \text{(where } \Delta x = (b-a)/n\text{)}\end{aligned}$$

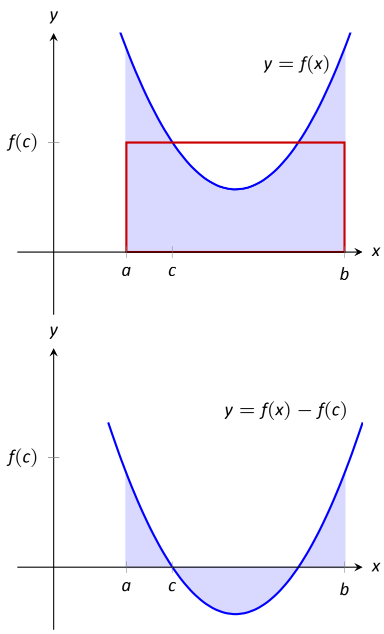


Figure 1.28: On top, a graph of $y = f(x)$ and the rectangle guaranteed by the Mean Value Theorem. Below, $y = f(x)$ is shifted down by $f(c)$; the resulting "area under the curve" is 0.

Notes:

Now take the limit as $n \to \infty$:

$$\lim_{n\to\infty} \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\Delta x \quad = \quad \frac{1}{b-a}\int_a^b f(x)\ dx \quad = \quad f(c).$$

This tells us this: when we evaluate f at n (somewhat) equally spaced points in $[a,b]$, the average value of these samples is $f(c)$ as $n \to \infty$.

This leads us to a definition.

Definition 1.4.1 The Average Value of f on $[a,b]$

Let f be continuous on $[a,b]$. The **average value of f on $[a,b]$** is $f(c)$, where c is a value in $[a,b]$ guaranteed by the Mean Value Theorem. I.e.,

$$\text{Average Value of } f \text{ on } [a,b] = \frac{1}{b-a}\int_a^b f(x)\ dx.$$

An application of this definition is given in the following example.

Example 1.4.9 Finding the average value of a function

An object moves back and forth along a straight line with a velocity given by $v(t) = (t-1)^2$ on $[0,3]$, where t is measured in seconds and $v(t)$ is measured in ft/s.

What is the average velocity of the object?

Solution By our definition, the average velocity is:

$$\frac{1}{3-0}\int_0^3 (t-1)^2\ dt = \frac{1}{3}\int_0^3 \left(t^2-2t+1\right)\ dt = \frac{1}{3}\left(\frac{1}{3}t^3 - t^2 + t\right)\Bigg|_0^3 = 1 \text{ ft/s}.$$

We can understand the above example through a simpler situation. Suppose you drove 100 miles in 2 hours. What was your average speed? The answer is simple: displacement/time = 100 miles/2 hours = 50 mph.

What was the displacement of the object in Example 1.4.9? We calculate this by integrating its velocity function: $\int_0^3 (t-1)^2\ dt = 3$ ft. Its final position was 3 feet from its initial position after 3 seconds: its average velocity was 1 ft/s.

This section has laid the groundwork for a lot of great mathematics to follow. The most important lesson is this: definite integrals can be evaluated using antiderivatives. Since the previous section established that definite integrals are the limit of Riemann sums, we can later create

Notes:

Riemann sums to approximate values other than "area under the curve," convert the sums to definite integrals, then evaluate these using the Fundamental Theorem of Calculus. This will allow us to compute the work done by a variable force, the volume of certain solids, the arc length of curves, and more.

The downside is this: generally speaking, computing antiderivatives is much more difficult than computing derivatives. The next chapter is devoted to techniques of finding antiderivatives so that a wide variety of definite integrals can be evaluated. Before that, the next section explores techniques of approximating the value of definite integrals beyond using the Left Hand, Right Hand and Midpoint Rules.

Notes:

Exercises 1.4

Terms and Concepts

1. How are definite and indefinite integrals related?

2. What constant of integration is most commonly used when evaluating definite integrals?

3. T/F: If f is a continuous function, then $F(x) = \int_a^x f(t)\ dt$ is also a continuous function.

4. The definite integral can be used to find "the area under a curve." Give two other uses for definite integrals.

Problems

In Exercises 5 – 28, evaluate the definite integral.

5. $\int_1^3 (3x^2 - 2x + 1)\ dx$

6. $\int_0^4 (x-1)^2\ dx$

7. $\int_{-1}^1 (x^3 - x^5)\ dx$

8. $\int_{\pi/2}^{\pi} \cos x\ dx$

9. $\int_0^{\pi/4} \sec^2 x\ dx$

10. $\int_1^e \frac{1}{x}\ dx$

11. $\int_{-1}^1 5^x\ dx$

12. $\int_{-2}^{-1} (4 - 2x^3)\ dx$

13. $\int_0^{\pi} (2\cos x - 2\sin x)\ dx$

14. $\int_1^3 e^x\ dx$

15. $\int_0^4 \sqrt{t}\ dt$

16. $\int_9^{25} \frac{1}{\sqrt{t}}\ dt$

17. $\int_1^8 \sqrt[3]{x}\ dx$

18. $\int_1^2 \frac{1}{x}\ dx$

19. $\int_1^2 \frac{1}{x^2}\ dx$

20. $\int_1^2 \frac{1}{x^3}\ dx$

21. $\int_0^1 x\ dx$

22. $\int_0^1 x^2\ dx$

23. $\int_0^1 x^3\ dx$

24. $\int_0^1 x^{100}\ dx$

25. $\int_{-4}^4 dx$

26. $\int_{-10}^{-5} 3\ dx$

27. $\int_{-2}^2 0\ dx$

28. $\int_{\pi/6}^{\pi/3} \csc x \cot x\ dx$

29. Explain why:

    (a) $\int_{-1}^1 x^n\ dx = 0$, when n is a positive, odd integer, and

    (b) $\int_{-1}^1 x^n\ dx = 2\int_0^1 x^n\ dx$ when n is a positive, even integer.

In Exercises 30 – 33, find a value c guaranteed by the Mean Value Theorem.

30. $\int_0^2 x^2\ dx$

31. $\int_{-2}^2 x^2\ dx$

32. $\int_0^1 e^x\ dx$

33. $\int_0^{16} \sqrt{x}\ dx$

In Exercises 34 – 39, find the average value of the function on the given interval.

34. $f(x) = \sin x$ on $[0, \pi/2]$

35. $y = \sin x$ on $[0, \pi]$

36. $y = x$ on $[0, 4]$

37. $y = x^2$ on $[0, 4]$

38. $y = x^3$ on $[0, 4]$

39. $g(t) = 1/t$ on $[1, e]$

In Exercises 40 – 44, a velocity function of an object moving along a straight line is given. Find the displacement of the object over the given time interval.

40. $v(t) = -32t + 20$ft/s on $[0, 5]$

41. $v(t) = -32t + 200$ft/s on $[0, 10]$

42. $v(t) = 2^t$mph on $[-1, 1]$

43. $v(t) = \cos t$ ft/s on $[0, 3\pi/2]$

44. $v(t) = \sqrt[4]{t}$ ft/s on $[0, 16]$

In Exercises 45 – 48, an acceleration function of an object moving along a straight line is given. Find the change of the object's velocity over the given time interval.

45. $a(t) = -32$ft/s^2 on $[0, 2]$

46. $a(t) = 10$ft/s^2 on $[0, 5]$

47. $a(t) = t$ ft/s^2 on $[0, 2]$

48. $a(t) = \cos t$ ft/s^2 on $[0, \pi]$

In Exercises 49 – 52, sketch the given functions and find the area of the enclosed region.

49. $y = 2x$, $y = 5x$, and $x = 3$.

50. $y = -x + 1$, $y = 3x + 6$, $x = 2$ and $x = -1$.

51. $y = x^2 - 2x + 5$, $y = 5x - 5$.

52. $y = 2x^2 + 2x - 5$, $y = x^2 + 3x + 7$.

In Exercises 53 – 56, find $F'(x)$.

53. $F(x) = \displaystyle\int_2^{x^3+x} \frac{1}{t}\ dt$

54. $F(x) = \displaystyle\int_{x^3}^{0} t^3\ dt$

55. $F(x) = \displaystyle\int_{x}^{x^2} (t+2)\ dt$

56. $F(x) = \displaystyle\int_{\ln x}^{e^x} \sin t\ dt$

2: Techniques of Integration

The previous chapter introduced the antiderivative and connected it to signed areas under a curve through the Fundamental Theorem of Calculus. The next chapter explores more applications of definite integrals than just area. As evaluating definite integrals will become important, we will want to find antiderivatives of a variety of functions.

This chapter is devoted to exploring techniques of antidifferentiation. While not every function has an antiderivative in terms of elementary functions (a concept introduced in the section on Numerical Integration), we can still find antiderivatives of a wide variety of functions.

2.1 Substitution

We motivate this section with an example. Let $f(x) = (x^2 + 3x - 5)^{10}$. We can compute $f'(x)$ using the Chain Rule. It is:

$$f'(x) = 10(x^2 + 3x - 5)^9 \cdot (2x + 3) = (20x + 30)(x^2 + 3x - 5)^9.$$

Now consider this: What is $\int (20x + 30)(x^2 + 3x - 5)^9 \ dx$? We have the answer in front of us;

$$\int (20x + 30)(x^2 + 3x - 5)^9 \ dx = (x^2 + 3x - 5)^{10} + C.$$

How would we have evaluated this indefinite integral without starting with $f(x)$ as we did?

This section explores *integration by substitution.* It allows us to "undo the Chain Rule." Substitution allows us to evaluate the above integral without knowing the original function first.

The underlying principle is to rewrite a "complicated" integral of the form $\int f(x) \ dx$ as a not–so–complicated integral $\int h(u) \ du$. We'll formally establish later how this is done. First, consider again our introductory indefinite integral, $\int (20x + 30)(x^2 + 3x - 5)^9 \ dx$. Arguably the most "complicated" part of the integrand is $(x^2 + 3x - 5)^9$. We wish to make this simpler; we do so through a substitution. Let $u = x^2 + 3x - 5$. Thus

$$(x^2 + 3x - 5)^9 = u^9.$$

We have established u as a function of x, so now consider the differential of u:

$$du = (2x + 3)dx.$$

Keep in mind that $(2x + 3)$ and dx are multiplied; the dx is not "just sitting there."

Return to the original integral and do some substitutions through algebra:

$$\begin{aligned}
\int (20x + 30)(x^2 + 3x - 5)^9 \ dx &= \int 10(2x + 3)(x^2 + 3x - 5)^9 \ dx \\
&= \int 10(\underbrace{x^2 + 3x - 5}_{u})^9 \ \underbrace{(2x + 3) \ dx}_{du} \\
&= \int 10u^9 \ du \\
&= u^{10} + C \quad \text{(replace } u \text{ with } x^2 + 3x - 5\text{)} \\
&= (x^2 + 3x - 5)^{10} + C
\end{aligned}$$

One might well look at this and think "I (sort of) followed how that worked, but I could never come up with that on my own," but the process is learnable. This section contains numerous examples through which the reader will gain understanding and mathematical maturity enabling them to regard substitution as a natural tool when evaluating integrals.

We stated before that integration by substitution "undoes" the Chain Rule. Specifically, let $F(x)$ and $g(x)$ be differentiable functions and consider the derivative of their composition:

$$\frac{d}{dx}\Big(F\big(g(x)\big)\Big) = F'(g(x))g'(x).$$

Thus

$$\int F'(g(x))g'(x) \ dx = F(g(x)) + C.$$

Integration by substitution works by recognizing the "inside" function $g(x)$ and replacing it with a variable. By setting $u = g(x)$, we can rewrite the derivative as

$$\frac{d}{dx}\Big(F\big(u\big)\Big) = F'(u)u'.$$

Since $du = g'(x)dx$, we can rewrite the above integral as

$$\int F'(g(x))g'(x) \ dx = \int F'(u)du = F(u) + C = F(g(x)) + C.$$

This concept is important so we restate it in the context of a theorem.

Notes:

Theorem 2.1.1 Integration by Substitution

Let F and g be differentiable functions, where the range of g is an interval I contained in the domain of F. Then

$$\int F'(g(x))g'(x)\ dx = F(g(x)) + C.$$

If $u = g(x)$, then $du = g'(x)dx$ and

$$\int F'(g(x))g'(x)\ dx = \int F'(u)\ du = F(u) + C = F(g(x)) + C.$$

The point of substitution is to make the integration step easy. Indeed, the step $\int F'(u)\ du = F(u) + C$ looks easy, as the antiderivative of the derivative of F is just F, plus a constant. The "work" involved is making the proper substitution. There is not a step–by–step process that one can memorize; rather, experience will be one's guide. To gain experience, we now embark on many examples.

Example 2.1.1 Integrating by substitution

Evaluate $\displaystyle\int x\sin(x^2+5)\ dx$.

Solution Knowing that substitution is related to the Chain Rule, we choose to let u be the "inside" function of $\sin(x^2+5)$. (This is not *always* a good choice, but it is often the best place to start.)

Let $u = x^2 + 5$, hence $du = 2x\,dx$. The integrand has an $x\,dx$ term, but not a $2x\,dx$ term. (Recall that multiplication is commutative, so the x does not physically have to be next to dx for there to be an $x\,dx$ term.) We can divide both sides of the du expression by 2:

$$du = 2x\,dx \quad \Rightarrow \quad \frac{1}{2}du = x\,dx.$$

We can now substitute.

$$\begin{aligned}\int x\sin(x^2+5)\ dx &= \int \sin(\underbrace{x^2+5}_{u})\ \underbrace{x\ dx}_{\frac{1}{2}du}\\ &= \int \frac{1}{2}\sin u\ du\end{aligned}$$

Notes:

$$= -\frac{1}{2}\cos u + C \quad \text{(now replace } u \text{ with } x^2+5)$$
$$= -\frac{1}{2}\cos(x^2+5) + C.$$

Thus $\int x\sin(x^2+5)\ dx = -\frac{1}{2}\cos(x^2+5)+C$. We can check our work by evaluating the derivative of the right hand side.

Example 2.1.2 Integrating by substitution

Evaluate $\displaystyle\int \cos(5x)\ dx$.

Solution Again let u replace the "inside" function. Letting $u = 5x$, we have $du = 5dx$. Since our integrand does not have a $5dx$ term, we can divide the previous equation by 5 to obtain $\frac{1}{5}du = dx$. We can now substitute.

$$\int \cos(5x)\ dx = \int \cos(\underbrace{5x}_{u})\ \underbrace{dx}_{\frac{1}{5}du}$$
$$= \int \frac{1}{5}\cos u\ du$$
$$= \frac{1}{5}\sin u + C$$
$$= \frac{1}{5}\sin(5x) + C.$$

We can again check our work through differentiation.

The previous example exhibited a common, and simple, type of substitution. The "inside" function was a linear function (in this case, $y = 5x$). When the inside function is linear, the resulting integration is very predictable, outlined here.

Key Idea 2.1.1 Substitution With A Linear Function

Consider $\int F'(ax+b)\ dx$, where $a \neq 0$ and b are constants. Letting $u = ax+b$ gives $du = a\cdot dx$, leading to the result

$$\int F'(ax+b)\ dx = \frac{1}{a}F(ax+b)+C.$$

Notes:

Thus $\int \sin(7x-4)\ dx = -\frac{1}{7}\cos(7x-4)+C$. Our next example can use Key Idea 2.1.1, but we will only employ it after going through all of the steps.

Example 2.1.3 Integrating by substituting a linear function

Evaluate $\displaystyle\int \frac{7}{-3x+1}\ dx$.

Solution View this a composition of functions $f(g(x))$, where $f(x)=7/x$ and $g(x)=-3x+1$. Employing our understanding of substitution, we let $u=-3x+1$, the inside function. Thus $du=-3dx$. The integrand lacks a -3; hence divide the previous equation by -3 to obtain $-du/3=dx$. We can now evaluate the integral through substitution.

$$\begin{aligned}\int \frac{7}{-3x+1}\ dx &= \int \frac{7}{u}\frac{du}{-3}\\ &= \frac{-7}{3}\int \frac{du}{u}\\ &= \frac{-7}{3}\ln|u|+C\\ &= -\frac{7}{3}\ln|-3x+1|+C.\end{aligned}$$

Using Key Idea 2.1.1 is faster, recognizing that u is linear and $a=-3$. One may want to continue writing out all the steps until they are comfortable with this particular shortcut.

Not all integrals that benefit from substitution have a clear "inside" function. Several of the following examples will demonstrate ways in which this occurs.

Example 2.1.4 Integrating by substitution

Evaluate $\displaystyle\int \sin x\cos x\ dx$.

Solution There is not a composition of function here to exploit; rather, just a product of functions. Do not be afraid to experiment; when given an integral to evaluate, it is often beneficial to think "If I let u be *this*, then du must be *that* ..." and see if this helps simplify the integral at all.

In this example, let's set $u=\sin x$. Then $du=\cos x\ dx$, which we have

Notes:

as part of the integrand! The substitution becomes very straightforward:

$$\begin{aligned}\int \sin x \cos x \; dx &= \int u \; du \\ &= \frac{1}{2}u^2 + C \\ &= \frac{1}{2}\sin^2 x + C.\end{aligned}$$

One would do well to ask "What would happen if we let $u = \cos x$?" The result is just as easy to find, yet looks very different. The challenge to the reader is to evaluate the integral letting $u = \cos x$ and discover why the answer is the same, yet looks different.

Our examples so far have required "basic substitution." The next example demonstrates how substitutions can be made that often strike the new learner as being "nonstandard."

Example 2.1.5 Integrating by substitution

Evaluate $\displaystyle\int x\sqrt{x+3} \; dx$.

Solution Recognizing the composition of functions, set $u = x+3$. Then $du = dx$, giving what seems initially to be a simple substitution. But at this stage, we have:

$$\int x\sqrt{x+3} \; dx = \int x\sqrt{u} \; du.$$

We cannot evaluate an integral that has both an x and an u in it. We need to convert the x to an expression involving just u.

Since we set $u = x+3$, we can also state that $u - 3 = x$. Thus we can replace x in the integrand with $u - 3$. It will also be helpful to rewrite $\sqrt{u}$ as $u^{\frac{1}{2}}$.

$$\begin{aligned}\int x\sqrt{x+3} \; dx &= \int (u-3)u^{\frac{1}{2}} \; du \\ &= \int \left(u^{\frac{3}{2}} - 3u^{\frac{1}{2}}\right) \; du \\ &= \frac{2}{5}u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + C \\ &= \frac{2}{5}(x+3)^{\frac{5}{2}} - 2(x+3)^{\frac{3}{2}} + C.\end{aligned}$$

Notes:

Checking your work is always a good idea. In this particular case, some algebra will be needed to make one's answer match the integrand in the original problem.

Example 2.1.6 Integrating by substitution

Evaluate $\displaystyle\int \frac{1}{x \ln x}\ dx$.

Solution This is another example where there does not seem to be an obvious composition of functions. The line of thinking used in Example 2.1.5 is useful here: choose something for u and consider what this implies du must be. If u can be chosen such that du also appears in the integrand, then we have chosen well.

Choosing $u = 1/x$ makes $du = -1/x^2\ dx$; that does not seem helpful. However, setting $u = \ln x$ makes $du = 1/x\ dx$, which is part of the integrand. Thus:

$$\begin{aligned}
\int \frac{1}{x \ln x}\ dx &= \int \underbrace{\frac{1}{\ln x}}_{1/u} \underbrace{\frac{1}{x}\ dx}_{du} \\
&= \int \frac{1}{u}\ du \\
&= \ln |u| + C \\
&= \ln |\ln x| + C.
\end{aligned}$$

The final answer is interesting; the natural log of the natural log. Take the derivative to confirm this answer is indeed correct.

Integrals Involving Trigonometric Functions

Section 2.3 delves deeper into integrals of a variety of trigonometric functions; here we use substitution to establish a foundation that we will build upon.

The next three examples will help fill in some missing pieces of our antiderivative knowledge. We know the antiderivatives of the sine and cosine functions; what about the other standard functions tangent, cotangent, secant and cosecant? We discover these next.

Example 2.1.7 Integration by substitution: antiderivatives of $\tan x$

Evaluate $\displaystyle\int \tan x\ dx$.

Notes:

Solution The previous paragraph established that we did not know the antiderivatives of tangent, hence we must assume that we have learned something in this section that can help us evaluate this indefinite integral.

Rewrite $\tan x$ as $\sin x/\cos x$. While the presence of a composition of functions may not be immediately obvious, recognize that $\cos x$ is "inside" the $1/x$ function. Therefore, we see if setting $u = \cos x$ returns usable results. We have that $du = -\sin x\ dx$, hence $-du = \sin x\ dx$. We can integrate:

$$\begin{aligned}\int \tan x\ dx &= \int \frac{\sin x}{\cos x}\ dx \\ &= \int \underbrace{\frac{1}{\cos x}}_{u} \underbrace{\sin x\ dx}_{-du} \\ &= \int \frac{-1}{u}\ du \\ &= -\ln|u| + C \\ &= -\ln|\cos x| + C.\end{aligned}$$

Some texts prefer to bring the -1 inside the logarithm as a power of $\cos x$, as in:

$$\begin{aligned}-\ln|\cos x| + C &= \ln|(\cos x)^{-1}| + C \\ &= \ln\left|\frac{1}{\cos x}\right| + C \\ &= \ln|\sec x| + C.\end{aligned}$$

Thus the result they give is $\int \tan x\ dx = \ln|\sec x| + C$. These two answers are equivalent.

Example 2.1.8 Integrating by substitution: antiderivatives of $\sec x$

Evaluate $\displaystyle\int \sec x\ dx$.

Solution This example employs a wonderful trick: multiply the integrand by "1" so that we see how to integrate more clearly. In this case, we write "1" as

$$1 = \frac{\sec x + \tan x}{\sec x + \tan x}.$$

Notes:

This may seem like it came out of left field, but it works beautifully. Consider:

$$\begin{aligned}\int \sec x \; dx &= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \; dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \; dx.\end{aligned}$$

Now let $u = \sec x + \tan x$; this means $du = (\sec x \tan x + \sec^2 x) \; dx$, which is our numerator. Thus:

$$\begin{aligned}&= \int \frac{du}{u} \\ &= \ln |u| + C \\ &= \ln |\sec x + \tan x| + C.\end{aligned}$$

We can use similar techniques to those used in Examples 2.1.7 and 2.1.8 to find antiderivatives of $\cot x$ and $\csc x$ (which the reader can explore in the exercises.) We summarize our results here.

Theorem 2.1.2 Antiderivatives of Trigonometric Functions

1. $\int \sin x \; dx = -\cos x + C$
2. $\int \cos x \; dx = \sin x + C$
3. $\int \tan x \; dx = -\ln |\cos x| + C$
4. $\int \csc x \; dx = -\ln |\csc x + \cot x| + C$
5. $\int \sec x \; dx = \ln |\sec x + \tan x| + C$
6. $\int \cot x \; dx = \ln |\sin x| + C$

We explore one more common trigonometric integral.

Example 2.1.9 Integration by substitution: powers of $\cos x$ and $\sin x$

Evaluate $\int \cos^2 x \; dx$.

Solution We have a composition of functions as $\cos^2 x = \big(\cos x\big)^2$. However, setting $u = \cos x$ means $du = -\sin x \; dx$, which we do not have in the integral. Another technique is needed.

Notes:

The process we'll employ is to use a Power Reducing formula for $\cos^2 x$ (perhaps consult the back of this text for this formula), which states

$$\cos^2 x = \frac{1+\cos(2x)}{2}.$$

The right hand side of this equation is not difficult to integrate. We have:

$$\begin{aligned}\int \cos^2 x \; dx &= \int \frac{1+\cos(2x)}{2} \; dx \\ &= \int \left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right) \; dx.\end{aligned}$$

Now use Key Idea 2.1.1:

$$\begin{aligned}&= \frac{1}{2}x + \frac{1}{2}\frac{\sin(2x)}{2} + C \\ &= \frac{1}{2}x + \frac{\sin(2x)}{4} + C.\end{aligned}$$

We'll make significant use of this power–reducing technique in future sections.

Simplifying the Integrand

It is common to be reluctant to manipulate the integrand of an integral; at first, our grasp of integration is tenuous and one may think that working with the integrand will improperly change the results. Integration by substitution works using a different logic: as long as *equality* is maintained, the integrand can be manipulated so that its *form* is easier to deal with. The next two examples demonstrate common ways in which using algebra first makes the integration easier to perform.

Example 2.1.10 Integration by substitution: simplifying first

Evaluate $\displaystyle\int \frac{x^3+4x^2+8x+5}{x^2+2x+1} \; dx.$

Solution One may try to start by setting u equal to either the numerator or denominator; in each instance, the result is not workable.

When dealing with rational functions (i.e., quotients made up of polynomial functions), it is an almost universal rule that everything works better when the degree of the numerator is less than the degree of the denominator. Hence we use polynomial division.

Notes:

We skip the specifics of the steps, but note that when $x^2 + 2x + 1$ is divided into $x^3 + 4x^2 + 8x + 5$, it goes in $x + 2$ times with a remainder of $3x + 3$. Thus

$$\frac{x^3 + 4x^2 + 8x + 5}{x^2 + 2x + 1} = x + 2 + \frac{3x + 3}{x^2 + 2x + 1}.$$

Integrating $x + 2$ is simple. The fraction can be integrated by setting $u = x^2 + 2x + 1$, giving $du = (2x + 2)\ dx$. This is very similar to the numerator. Note that $du/2 = (x + 1)\ dx$ and then consider the following:

$$\begin{aligned}
\int \frac{x^3 + 4x^2 + 8x + 5}{x^2 + 2x + 1}\ dx &= \int \left(x + 2 + \frac{3x + 3}{x^2 + 2x + 1} \right)\ dx \\
&= \int (x + 2)\ dx + \int \frac{3(x + 1)}{x^2 + 2x + 1}\ dx \\
&= \frac{1}{2}x^2 + 2x + C_1 + \int \frac{3}{u}\frac{du}{2} \\
&= \frac{1}{2}x^2 + 2x + C_1 + \frac{3}{2}\ln|u| + C_2 \\
&= \frac{1}{2}x^2 + 2x + \frac{3}{2}\ln|x^2 + 2x + 1| + C.
\end{aligned}$$

In some ways, we "lucked out" in that after dividing, substitution was able to be done. In later sections we'll develop techniques for handling rational functions where substitution is not directly feasible.

Example 2.1.11 Integration by alternate methods

Evaluate $\displaystyle\int \frac{x^2 + 2x + 3}{\sqrt{x}}\ dx$ with, and without, substitution.

Solution We already know how to integrate this particular example. Rewrite $\sqrt{x}$ as $x^{\frac{1}{2}}$ and simplify the fraction:

$$\frac{x^2 + 2x + 3}{x^{1/2}} = x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}.$$

We can now integrate using the Power Rule:

$$\begin{aligned}
\int \frac{x^2 + 2x + 3}{x^{1/2}}\ dx &= \int \left(x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + 3x^{-\frac{1}{2}} \right)\ dx \\
&= \frac{2}{5}x^{\frac{5}{2}} + \frac{4}{3}x^{\frac{3}{2}} + 6x^{\frac{1}{2}} + C
\end{aligned}$$

Notes:

This is a perfectly fine approach. We demonstrate how this can also be solved using substitution as its implementation is rather clever.

Let $u = \sqrt{x} = x^{\frac{1}{2}}$; therefore

$$du = \frac{1}{2}x^{-\frac{1}{2}}dx = \frac{1}{2\sqrt{x}}\ dx \quad \Rightarrow \quad 2du = \frac{1}{\sqrt{x}}\ dx.$$

This gives us $\int \frac{x^2+2x+3}{\sqrt{x}}\ dx = \int (x^2+2x+3)\cdot 2\ du$. What are we to do with the other x terms? Since $u = x^{\frac{1}{2}}$, $u^2 = x$, etc. We can then replace x^2 and x with appropriate powers of u. We thus have

$$\begin{aligned}\int \frac{x^2+2x+3}{\sqrt{x}}\ dx &= \int (x^2+2x+3)\cdot 2\ du \\ &= \int 2(u^4+2u^2+3)\ du \\ &= \frac{2}{5}u^5 + \frac{4}{3}u^3 + 6u + C \\ &= \frac{2}{5}x^{\frac{5}{2}} + \frac{4}{3}x^{\frac{3}{2}} + 6x^{\frac{1}{2}} + C,\end{aligned}$$

which is obviously the same answer we obtained before. In this situation, substitution is arguably more work than our other method. The fantastic thing is that it works. It demonstrates how flexible integration is.

Substitution and Inverse Trigonometric Functions

When studying derivatives of inverse functions, we learned that

$$\frac{d}{dx}\big(\tan^{-1} x\big) = \frac{1}{1+x^2}.$$

Applying the Chain Rule to this is not difficult; for instance,

$$\frac{d}{dx}\big(\tan^{-1} 5x\big) = \frac{5}{1+25x^2}.$$

We now explore how Substitution can be used to "undo" certain derivatives that are the result of the Chain Rule applied to Inverse Trigonometric functions. We begin with an example.

Example 2.1.12 Integrating by substitution: inverse trigonometric functions

Evaluate $\int \frac{1}{25+x^2}\ dx$.

Notes:

Solution The integrand looks similar to the derivative of the arctangent function. Note:

$$\begin{aligned}\frac{1}{25+x^2} &= \frac{1}{25(1+\frac{x^2}{25})}\\ &= \frac{1}{25(1+\left(\frac{x}{5}\right)^2)}\\ &= \frac{1}{25}\frac{1}{1+\left(\frac{x}{5}\right)^2}.\end{aligned}$$

Thus

$$\int \frac{1}{25+x^2}\ dx = \frac{1}{25}\int \frac{1}{1+\left(\frac{x}{5}\right)^2}\ dx.$$

This can be integrated using Substitution. Set $u = x/5$, hence $du = dx/5$ or $dx = 5du$. Thus

$$\begin{aligned}\int \frac{1}{25+x^2}\ dx &= \frac{1}{25}\int \frac{1}{1+\left(\frac{x}{5}\right)^2}\ dx\\ &= \frac{1}{5}\int \frac{1}{1+u^2}\ du\\ &= \frac{1}{5}\tan^{-1} u + C\\ &= \frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) + C\end{aligned}$$

Example 2.1.12 demonstrates a general technique that can be applied to other integrands that result in inverse trigonometric functions. The results are summarized here.

Theorem 2.1.3 Integrals Involving Inverse Trigonomentric Functions

Let $a > 0$.

1. $\displaystyle\int \frac{1}{a^2+x^2}\ dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
2. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\ dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
3. $\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}}\ dx = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) + C$

Notes:

Let's practice using Theorem 2.1.3.

Example 2.1.13 Integrating by substitution: inverse trigonometric functions
Evaluate the given indefinite integrals.

$$\int \frac{1}{9+x^2}\,dx, \quad \int \frac{1}{x\sqrt{x^2-\frac{1}{100}}}\,dx \quad \text{and} \quad \int \frac{1}{\sqrt{5-x^2}}\,dx.$$

Solution Each can be answered using a straightforward application of Theorem 2.1.3.

$$\int \frac{1}{9+x^2}\,dx = \frac{1}{3}\tan^{-1}\frac{x}{3} + C, \text{ as } a = 3.$$

$$\int \frac{1}{x\sqrt{x^2-\frac{1}{100}}}\,dx = 10\sec^{-1}10x + C, \text{ as } a = \tfrac{1}{10}.$$

$$\int \frac{1}{\sqrt{5-x^2}} = \sin^{-1}\frac{x}{\sqrt{5}} + C, \text{ as } a = \sqrt{5}.$$

Most applications of Theorem 2.1.3 are not as straightforward. The next examples show some common integrals that can still be approached with this theorem.

Example 2.1.14 Integrating by substitution: completing the square
Evaluate $\displaystyle\int \frac{1}{x^2-4x+13}\,dx$.

Solution Initially, this integral seems to have nothing in common with the integrals in Theorem 2.1.3. As it lacks a square root, it almost certainly is not related to arcsine or arcsecant. It is, however, related to the arctangent function.

We see this by *completing the square* in the denominator. We give a brief reminder of the process here.

Start with a quadratic with a leading coefficient of 1. It will have the form of $x^2 + bx + c$. Take 1/2 of b, square it, and add/subtract it back

Notes:

into the expression. I.e.,

$$\begin{aligned} x^2+bx+c &= \underbrace{x^2+bx+\frac{b^2}{4}}_{(x+b/2)^2} - \frac{b^2}{4} + c \\ &= \left(x+\frac{b}{2}\right)^2 + c - \frac{b^2}{4} \end{aligned}$$

In our example, we take half of -4 and square it, getting 4. We add/subtract it into the denominator as follows:

$$\begin{aligned} \frac{1}{x^2-4x+13} &= \frac{1}{\underbrace{x^2-4x+4}_{(x-2)^2}-4+13} \\ &= \frac{1}{(x-2)^2+9} \end{aligned}$$

We can now integrate this using the arctangent rule. Technically, we need to substitute first with $u = x-2$, but we can employ Key Idea 2.1.1 instead. Thus we have

$$\int \frac{1}{x^2-4x+13}\,dx = \int \frac{1}{(x-2)^2+9}\,dx = \frac{1}{3}\tan^{-1}\frac{x-2}{3} + C.$$

Example 2.1.15 Integrals requiring multiple methods

Evaluate $\displaystyle\int \frac{4-x}{\sqrt{16-x^2}}\,dx$.

Solution This integral requires two different methods to evaluate it. We get to those methods by splitting up the integral:

$$\int \frac{4-x}{\sqrt{16-x^2}}\,dx = \int \frac{4}{\sqrt{16-x^2}}\,dx - \int \frac{x}{\sqrt{16-x^2}}\,dx.$$

The first integral is handled using a straightforward application of Theorem 2.1.3; the second integral is handled by substitution, with $u = 16-x^2$. We handle each separately.

$\displaystyle\int \frac{4}{\sqrt{16-x^2}}\,dx = 4\sin^{-1}\frac{x}{4} + C.$

$\displaystyle\int \frac{x}{\sqrt{16-x^2}}\,dx$: Set $u = 16-x^2$, so $du = -2xdx$ and $xdx = -du/2$.

Notes:

We have

$$\begin{aligned}\int \frac{x}{\sqrt{16-x^2}}\,dx &= \int \frac{-du/2}{\sqrt{u}} \\ &= -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du \\ &= -\sqrt{u}+C \\ &= -\sqrt{16-x^2}+C.\end{aligned}$$

Combining these together, we have

$$\int \frac{4-x}{\sqrt{16-x^2}}\,dx = 4\sin^{-1}\frac{x}{4} + \sqrt{16-x^2} + C.$$

Substitution and Definite Integration

This section has focused on evaluating indefinite integrals as we are learning a new technique for finding antiderivatives. However, much of the time integration is used in the context of a definite integral. Definite integrals that require substitution can be calculated using the following workflow:

1. Start with a definite integral $\displaystyle\int_a^b f(x)\,dx$ that requires substitution.
2. Ignore the bounds; use substitution to evaluate $\displaystyle\int f(x)\,dx$ and find an antiderivative $F(x)$.
3. Evaluate $F(x)$ at the bounds; that is, evaluate $F(x)\Big|_a^b = F(b)-F(a)$.

This workflow works fine, but substitution offers an alternative that is powerful and amazing (and a little time saving).

At its heart, (using the notation of Theorem 2.1.1) substitution converts integrals of the form $\int F'(g(x))g'(x)\,dx$ into an integral of the form $\int F'(u)\,du$ with the substitution of $u=g(x)$. The following theorem states how the bounds of a definite integral can be changed as the substitution is performed.

Notes:

Theorem 2.1.4 Substitution with Definite Integrals

Let F and g be differentiable functions, where the range of g is an interval I that is contained in the domain of F. Then

$$\int_a^b F'\big(g(x)\big)g'(x)\ dx = \int_{g(a)}^{g(b)} F'(u)\ du.$$

In effect, Theorem 2.1.4 states that once you convert to integrating with respect to u, you do not need to switch back to evaluating with respect to x. A few examples will help one understand.

Example 2.1.16 Definite integrals and substitution: changing the bounds

Evaluate $\displaystyle\int_0^2 \cos(3x-1)\ dx$ using Theorem 2.1.4.

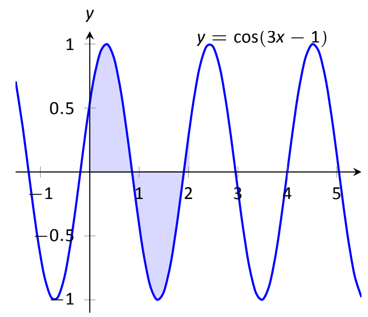
Solution Observing the composition of functions, let $u = 3x-1$, hence $du = 3dx$. As $3dx$ does not appear in the integrand, divide the latter equation by 3 to get $du/3 = dx$.

By setting $u = 3x-1$, we are implicitly stating that $g(x) = 3x-1$. Theorem 2.1.4 states that the new lower bound is $g(0) = -1$; the new upper bound is $g(2) = 5$. We now evaluate the definite integral:

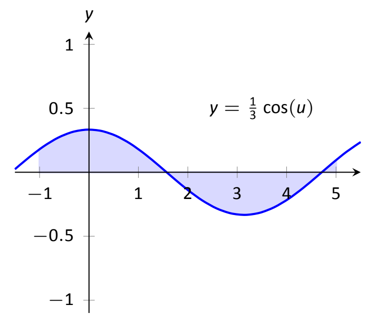
$$\begin{aligned}\int_1^2 \cos(3x-1)\ dx &= \int_{-1}^5 \cos u \frac{du}{3}\\ &= \frac{1}{3}\sin u\Big|_{-1}^5\\ &= \frac{1}{3}\big(\sin 5 - \sin(-1)\big) \approx -0.039.\end{aligned}$$

Notice how once we converted the integral to be in terms of u, we never went back to using x.

The graphs in Figure 2.1 tell more of the story. In (a) the area defined by the original integrand is shaded, whereas in (b) the area defined by the new integrand is shaded. In this particular situation, the areas look very similar; the new region is "shorter" but "wider," giving the same area.



(a)



(b)

Figure 2.1: Graphing the areas defined by the definite integrals of Example 2.1.16.

Example 2.1.17 Definite integrals and substitution: changing the bounds

Evaluate $\displaystyle\int_0^{\pi/2} \sin x \cos x\ dx$ using Theorem 2.1.4.

Notes:

Solution We saw the corresponding indefinite integral in Example 2.1.4. In that example we set $u = \sin x$ but stated that we could have let $u = \cos x$. For variety, we do the latter here.

Let $u = g(x) = \cos x$, giving $du = -\sin x\ dx$ and hence $\sin x\ dx = -du$. The new upper bound is $g(\pi/2) = 0$; the new lower bound is $g(0) = 1$. Note how the lower bound is actually larger than the upper bound now. We have

$$\begin{aligned}\int_0^{\pi/2} \sin x \cos x\ dx &= \int_1^0 -u\ du \quad \text{(switch bounds \& change sign)}\\ &= \int_0^1 u\ du\\ &= \frac{1}{2}u^2\Big|_0^1 = 1/2.\end{aligned}$$

In Figure 2.2 we have again graphed the two regions defined by our definite integrals. Unlike the previous example, they bear no resemblance to each other. However, Theorem 2.1.4 guarantees that they have the same area.

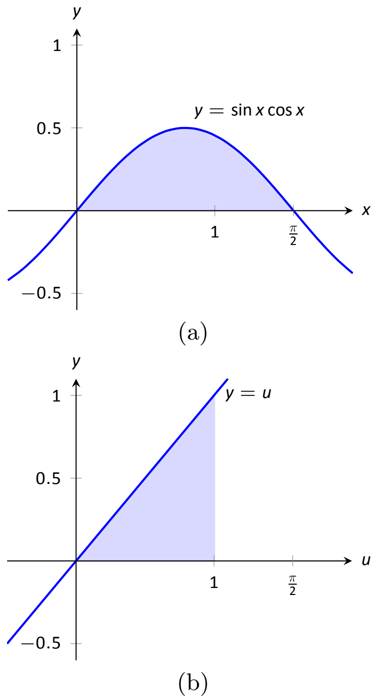


Figure 2.2: Graphing the areas defined by the definite integrals of Example 2.1.17.

Integration by substitution is a powerful and useful integration technique. The next section introduces another technique, called Integration by Parts. As substitution "undoes" the Chain Rule, integration by parts "undoes" the Product Rule. Together, these two techniques provide a strong foundation on which most other integration techniques are based.

Notes:

Exercises 2.1

Terms and Concepts

1. Substitution "undoes" what derivative rule?

2. T/F: One can use algebra to rewrite the integrand of an integral to make it easier to evaluate.

Problems

In Exercises 3 – 14, evaluate the indefinite integral to develop an understanding of Substitution.

3. $\int 3x^2\left(x^3-5\right)^7 dx$

4. $\int (2x-5)\left(x^2-5x+7\right)^3 dx$

5. $\int x\left(x^2+1\right)^8 dx$

6. $\int (12x+14)\left(3x^2+7x-1\right)^5 dx$

7. $\int \frac{1}{2x+7}dx$

8. $\int \frac{1}{\sqrt{2x+3}}dx$

9. $\int \frac{x}{\sqrt{x+3}}dx$

10. $\int \frac{x^3-x}{\sqrt{x}}dx$

11. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

12. $\int \frac{x^4}{\sqrt{x^5+1}}dx$

13. $\int \frac{\frac{1}{x}+1}{x^2}dx$

14. $\int \frac{\ln(x)}{x}dx$

In Exercises 15 – 23, use Substitution to evaluate the indefinite integral involving trigonometric functions.

15. $\int \sin^2(x)\cos(x)dx$

16. $\int \cos(3-6x)dx$

17. $\int \sec^2(4-x)dx$

18. $\int \sec(2x)dx$

19. $\int \tan^2(x)\sec^2(x)dx$

20. $\int x\cos\left(x^2\right)dx$

21. $\int \tan^2(x)dx$

22. $\int \cot x\ dx$. Do not just refer to Theorem 2.1.2 for the answer; justify it through Substitution.

23. $\int \csc x\ dx$. Do not just refer to Theorem 2.1.2 for the answer; justify it through Substitution.

In Exercises 24 – 30, use Substitution to evaluate the indefinite integral involving exponential functions.

24. $\int e^{3x-1}dx$

25. $\int e^{x^3}x^2dx$

26. $\int e^{x^2-2x+1}(x-1)dx$

27. $\int \frac{e^x+1}{e^x}dx$

28. $\int \frac{e^x-e^{-x}}{e^{2x}}dx$

29. $\int 3^{3x}dx$

30. $\int 4^{2x}dx$

In Exercises 31 – 34, use Substitution to evaluate the indefinite integral involving logarithmic functions.

31. $\int \frac{\ln x}{x}dx$

32. $\int \frac{\left(\ln x\right)^2}{x}dx$

33. $\int \frac{\ln\left(x^3\right)}{x}dx$

34. $\int \frac{1}{x\ln(x^2)}dx$

In Exercises 35 – 40, use Substitution to evaluate the indefinite integral involving rational functions.

35. $\int \frac{x^2+3x+1}{x}dx$

36. $\int \frac{x^3+x^2+x+1}{x}dx$

37. $\int \frac{x^3-1}{x+1}dx$

38. $\int \frac{x^2+2x-5}{x-3}dx$

39. $\int \frac{3x^2-5x+7}{x+1}dx$

40. $\int \frac{x^2+2x+1}{x^3+3x^2+3x}dx$

In Exercises 41 – 50, use Substitution to evaluate the indefinite integral involving inverse trigonometric functions.

41. $\int \frac{7}{x^2+7}dx$

42. $\int \frac{3}{\sqrt{9-x^2}}dx$

43. $\int \frac{14}{\sqrt{5-x^2}}dx$

44. $\int \frac{2}{x\sqrt{x^2-9}}dx$

45. $\int \frac{5}{\sqrt{x^4-16x^2}}dx$

46. $\int \frac{x}{\sqrt{1-x^4}}dx$

47. $\int \frac{1}{x^2-2x+8}dx$

48. $\int \frac{2}{\sqrt{-x^2+6x+7}}dx$

49. $\int \frac{3}{\sqrt{-x^2+8x+9}}dx$

50. $\int \frac{5}{x^2+6x+34}dx$

In Exercises 51 – 75, evaluate the indefinite integral.

51. $\int \frac{x^2}{(x^3+3)^2}dx$

52. $\int (3x^2+2x)\left(5x^3+5x^2+2\right)^8 dx$

53. $\int \frac{x}{\sqrt{1-x^2}}dx$

54. $\int x^2\csc^2\left(x^3+1\right)dx$

55. $\int \sin(x)\sqrt{\cos(x)}dx$

56. $\int \frac{1}{x-5}dx$

57. $\int \frac{7}{3x+2}dx$

58. $\int \frac{3x^3+4x^2+2x-22}{x^2+3x+5}dx$

59. $\int \frac{2x+7}{x^2+7x+3}dx$

60. $\int \frac{9(2x+3)}{3x^2+9x+7}dx$

61. $\int \frac{-x^3+14x^2-46x-7}{x^2-7x+1}dx$

62. $\int \frac{x}{x^4+81}dx$

63. $\int \frac{2}{4x^2+1}dx$

64. $\int \frac{1}{x\sqrt{4x^2-1}}dx$

65. $\int \frac{1}{\sqrt{16-9x^2}}dx$

66. $\int \frac{3x-2}{x^2-2x+10}dx$

67. $\int \frac{7-2x}{x^2+12x+61}dx$

68. $\int \frac{x^2+5x-2}{x^2-10x+32}dx$

69. $\int \frac{x^3}{x^2+9}dx$

70. $\int \frac{x^3-x}{x^2+4x+9}dx$

71. $\int \frac{\sin(x)}{\cos^2(x)+1}dx$

72. $\displaystyle\int \frac{\cos(x)}{\sin^2(x)+1}dx$

73. $\displaystyle\int \frac{\cos(x)}{1-\sin^2(x)}dx$

74. $\displaystyle\int \frac{3x-3}{\sqrt{x^2-2x-6}}dx$

75. $\displaystyle\int \frac{x-3}{\sqrt{x^2-6x+8}}dx$

In Exercises 76 – 83, evaluate the definite integral.

76. $\displaystyle\int_1^3 \frac{1}{x-5}dx$

77. $\displaystyle\int_2^6 x\sqrt{x-2}dx$

78. $\displaystyle\int_{-\pi/2}^{\pi/2} \sin^2 x\cos x\ dx$

79. $\displaystyle\int_0^1 2x(1-x^2)^4\ dx$

80. $\displaystyle\int_{-2}^{-1} (x+1)e^{x^2+2x+1}\ dx$

81. $\displaystyle\int_{-1}^1 \frac{1}{1+x^2}\ dx$

82. $\displaystyle\int_2^4 \frac{1}{x^2-6x+10}\ dx$

83. $\displaystyle\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\ dx$

2.2 Integration by Parts

Here's a simple integral that we can't yet evaluate:

$$\int x \cos x \, dx.$$

It's a simple matter to take the derivative of the integrand using the Product Rule, but there is no Product Rule for integrals. However, this section introduces *Integration by Parts*, a method of integration that is based on the Product Rule for derivatives. It will enable us to evaluate this integral.

The Product Rule says that if u and v are functions of x, then $(uv)' = u'v + uv'$. For simplicity, we've written u for $u(x)$ and v for $v(x)$. Suppose we integrate both sides with respect to x. This gives

$$\int (uv)' \, dx = \int (u'v + uv') \, dx.$$

By the Fundamental Theorem of Calculus, the left side integrates to uv. The right side can be broken up into two integrals, and we have

$$uv = \int u'v \, dx + \int uv' \, dx.$$

Solving for the second integral we have

$$\int uv' \, dx = uv - \int u'v \, dx.$$

Using differential notation, we can write $du = u'(x)dx$ and $dv = v'(x)dx$ and the expression above can be written as follows:

$$\int u \, dv = uv - \int v \, du.$$

This is the Integration by Parts formula. For reference purposes, we state this in a theorem.

Notes:

Theorem 2.2.1 Integration by Parts

Let u and v be differentiable functions of x on an interval I containing a and b. Then

$$\int u\ dv = uv - \int v\ du,$$

and

$$\int_{x=a}^{x=b} u\ dv = uv\Big|_a^b - \int_{x=a}^{x=b} v\ du.$$

Let's try an example to understand our new technique.

Example 2.2.1 Integrating using Integration by Parts

Evaluate $\displaystyle\int x \cos x\ dx$.

Solution The key to Integration by Parts is to identify part of the integrand as "u" and part as "dv." Regular practice will help one make good identifications, and later we will introduce some principles that help. For now, let $u = x$ and $dv = \cos x\ dx$.

It is generally useful to make a small table of these values as done below. Right now we only know u and dv as shown on the left of Figure 2.3; on the right we fill in the rest of what we need. If $u = x$, then $du = dx$. Since $dv = \cos x\ dx$, v is an antiderivative of $\cos x$. We choose $v = \sin x$.

$u = x$	$v = ?$		$u = x$	$v = \sin x$
$du = ?$	$dv = \cos x\ dx$	$\Rightarrow$	$du = dx$	$dv = \cos x\ dx$

Figure 2.3: Setting up Integration by Parts.

Now substitute all of this into the Integration by Parts formula, giving

$$\int x \cos x\, dx = x \sin x - \int \sin x\, dx.$$

We can then integrate $\sin x$ to get $-\cos x + C$ and overall our answer is

$$\int x \cos x\ dx = x \sin x + \cos x + C.$$

Note how the antiderivative contains a product, $x \sin x$. This product is what makes Integration by Parts necessary.

Notes:

The example above demonstrates how Integration by Parts works in general. We try to identify u and dv in the integral we are given, and the key is that we usually want to choose u and dv so that du is simpler than u and v is hopefully not too much more complicated than dv. This will mean that the integral on the right side of the Integration by Parts formula, $\int v\,du$ will be simpler to integrate than the original integral $\int u\,dv$.

In the example above, we chose $u = x$ and $dv = \cos x\,dx$. Then $du = dx$ was simpler than u and $v = \sin x$ is no more complicated than dv. Therefore, instead of integrating $x \cos x\,dx$, we could integrate $\sin x\,dx$, which we knew how to do.

A useful mnemonic for helping to determine u is "LIATE," where

L = **L**ogarithmic, I = **I**nverse Trig., A = **A**lgebraic (polynomials),
T = **T**rigonometric, and E = **E**xponential.

If the integrand contains both a logarithmic and an algebraic term, in general letting u be the logarithmic term works best, as indicated by L coming before A in LIATE.

We now consider another example.

Example 2.2.2 Integrating using Integration by Parts

Evaluate $\displaystyle\int xe^x\,dx$.

Solution The integrand contains an **A**lgebraic term (x) and an **E**xponential term (e^x). Our mnemonic suggests letting u be the algebraic term, so we choose $u = x$ and $dv = e^x\,dx$. Then $du = dx$ and $v = e^x$ as indicated by the tables below.

$$\begin{array}{ll} u = x & v = ? \\ du = ? & dv = e^x\ dx \end{array} \quad \Rightarrow \quad \begin{array}{ll} u = x & v = e^x \\ du = dx & dv = e^x\ dx \end{array}$$

Figure 2.4: Setting up Integration by Parts.

We see du is simpler than u, while there is no change in going from dv to v. This is good. The Integration by Parts formula gives

$$\int xe^x\,dx = xe^x - \int e^x\,dx.$$

The integral on the right is simple; our final answer is

$$\int xe^x\ dx = xe^x - e^x + C.$$

Notes:

Note again how the antiderivatives contain a product term.

Example 2.2.3 Integrating using Integration by Parts

Evaluate $\int x^2 \cos x \, dx$.

Solution The mnemonic suggests letting $u = x^2$ instead of the trigonometric function, hence $dv = \cos x \, dx$. Then $du = 2x \, dx$ and $v = \sin x$ as shown below.

$$\begin{array}{cc} u = x^2 & v = ? \\ du = ? & dv = \cos x \; dx \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = x^2 & v = \sin x \\ du = 2x \; dx & dv = \cos x \; dx \end{array}$$

Figure 2.5: Setting up Integration by Parts.

The Integration by Parts formula gives

$$\int x^2 \cos x \, dx = x^2 \sin x - \int 2x \sin x \, dx.$$

At this point, the integral on the right is indeed simpler than the one we started with, but to evaluate it, we need to do Integration by Parts again. Here we choose $u = 2x$ and $dv = \sin x$ and fill in the rest below.

$$\begin{array}{cc} u = 2x & v = ? \\ du = ? & dv = \sin x \; dx \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = 2x & v = -\cos x \\ du = 2 \; dx & dv = \sin x \; dx \end{array}$$

Figure 2.6: Setting up Integration by Parts (again).

$$\int x^2 \cos x \, dx = x^2 \sin x - \left(-2x \cos x - \int -2 \cos x \, dx \right).$$

The integral all the way on the right is now something we can evaluate. It evaluates to $-2 \sin x$. Then going through and simplifying, being careful to keep all the signs straight, our answer is

$$\int x^2 \cos x \; dx = x^2 \sin x + 2x \cos x - 2 \sin x + C.$$

Example 2.2.4 Integrating using Integration by Parts

Evaluate $\int e^x \cos x \, dx$.

Notes:

Solution This is a classic problem. Our mnemonic suggests letting u be the trigonometric function instead of the exponential. In this particular example, one can let u be either $\cos x$ or e^x; to demonstrate that we do not have to follow LIATE, we choose $u = e^x$ and hence $dv = \cos x\, dx$. Then $du = e^x\, dx$ and $v = \sin x$ as shown below.

$$\begin{array}{ll} u = e^x & v = ? \\ du = ? & dv = \cos x\ dx \end{array} \quad \Rightarrow \quad \begin{array}{ll} u = e^x & v = \sin x \\ du = e^x\ dx & dv = \cos x\ dx \end{array}$$

Figure 2.7: Setting up Integration by Parts.

Notice that du is no simpler than u, going against our general rule (but bear with us). The Integration by Parts formula yields

$$\int e^x \cos x\ dx = e^x \sin x - \int e^x \sin x\, dx.$$

The integral on the right is not much different than the one we started with, so it seems like we have gotten nowhere. Let's keep working and apply Integration by Parts to the new integral, using $u = e^x$ and $dv = \sin x\, dx$. This leads us to the following:

$$\begin{array}{ll} u = e^x & v = ? \\ du = ? & dv = \sin x\ dx \end{array} \quad \Rightarrow \quad \begin{array}{ll} u = e^x & v = -\cos x \\ du = e^x\ dx & dv = \sin x\ dx \end{array}$$

Figure 2.8: Setting up Integration by Parts (again).

The Integration by Parts formula then gives:

$$\begin{aligned} \int e^x \cos x\, dx &= e^x \sin x - \left(-e^x \cos x - \int -e^x \cos x\, dx\right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x\ dx. \end{aligned}$$

It seems we are back right where we started, as the right hand side contains $\int e^x \cos x\, dx$. But this is actually a good thing.

Add $\displaystyle\int e^x \cos x\ dx$ to both sides. This gives

$$2\int e^x \cos x\ dx = e^x \sin x + e^x \cos x$$

Notes:

Now divide both sides by 2:

$$\int e^x \cos x \; dx = \frac{1}{2}\big(e^x \sin x + e^x \cos x\big).$$

Simplifying a little and adding the constant of integration, our answer is thus

$$\int e^x \cos x \; dx = \frac{1}{2}e^x \left(\sin x + \cos x\right) + C.$$

Example 2.2.5 Integrating using Integration by Parts: antiderivative of $\ln x$
Evaluate $\displaystyle\int \ln x \, dx$.

Solution One may have noticed that we have rules for integrating the familiar trigonometric functions and e^x, but we have not yet given a rule for integrating $\ln x$. That is because $\ln x$ can't easily be integrated with any of the rules we have learned up to this point. But we can find its antiderivative by a clever application of Integration by Parts. Set $u = \ln x$ and $dv = dx$. This is a good, sneaky trick to learn as it can help in other situations. This determines $du = (1/x)\,dx$ and $v = x$ as shown below.

$$\begin{array}{ll} u = \ln x & v = \,? \\ du = \,? & dv = dx \end{array} \quad \Rightarrow \quad \begin{array}{ll} u = \ln x & v = x \\ du = 1/x \; dx & dv = dx \end{array}$$

Figure 2.9: Setting up Integration by Parts.

Putting this all together in the Integration by Parts formula, things work out very nicely:

$$\int \ln x \, dx = x \ln x - \int x \, \frac{1}{x} \, dx.$$

The new integral simplifies to $\int 1 \, dx$, which is about as simple as things get. Its integral is $x + C$ and our answer is

$$\int \ln x \; dx = x \ln x - x + C.$$

Example 2.2.6 Integrating using Int. by Parts: antiderivative of $\arctan x$
Evaluate $\displaystyle\int \arctan x \, dx$.

Notes:

Solution The same sneaky trick we used above works here. Let $u = \arctan x$ and $dv = dx$. Then $du = 1/(1+x^2)\,dx$ and $v = x$. The Integration by Parts formula gives

$$\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}\,dx.$$

The integral on the right can be solved by substitution. Taking $u = 1+x^2$, we get $du = 2x\,dx$. The integral then becomes

$$\int \arctan x\,dx = x\arctan x - \frac{1}{2}\int \frac{1}{u}\,du.$$

The integral on the right evaluates to $\ln|u| + C$, which becomes $\ln(1+x^2)+C$. Therefore, the answer is

$$\int \arctan x\;dx = x\arctan x - \ln(1+x^2) + C.$$

Notes:

Substitution Before Integration

When taking derivatives, it was common to employ multiple rules (such as using both the Quotient and the Chain Rules). It should then come as no surprise that some integrals are best evaluated by combining integration techniques. In particular, here we illustrate making an "unusual" substitution first before using Integration by Parts.

Example 2.2.7 Integration by Parts after substitution
Evaluate $\int \cos(\ln x)\ dx$.

Solution The integrand contains a composition of functions, leading us to think Substitution would be beneficial. Letting $u = \ln x$, we have $du = 1/x\ dx$. This seems problematic, as we do not have a $1/x$ in the integrand. But consider:

$$du = \frac{1}{x}\ dx \Rightarrow x \cdot du = dx.$$

Since $u = \ln x$, we can use inverse functions and conclude that $x = e^u$. Therefore we have that

$$\begin{aligned} dx &= x \cdot du \\ &= e^u\ du. \end{aligned}$$

We can thus replace $\ln x$ with u and dx with $e^u\ du$. Thus we rewrite our integral as

$$\int \cos(\ln x)\ dx = \int e^u \cos u\ du.$$

We evaluated this integral in Example 2.2.4. Using the result there, we have:

$$\begin{aligned} \int \cos(\ln x)\ dx &= \int e^u \cos u\ du \\ &= \frac{1}{2}e^u\big(\sin u + \cos u\big) + C \\ &= \frac{1}{2}e^{\ln x}\big(\sin(\ln x) + \cos(\ln x)\big) + C \\ &= \frac{1}{2}x\big(\sin(\ln x) + \cos(\ln x)\big) + C. \end{aligned}$$

Definite Integrals and Integration By Parts

Notes:

So far we have focused only on evaluating indefinite integrals. Of course, we can use Integration by Parts to evaluate definite integrals as well, as Theorem 2.2.1 states. We do so in the next example.

Example 2.2.8 Definite integration using Integration by Parts

Evaluate $\displaystyle\int_1^2 x^2 \ln x\, dx$.

Solution Our mnemonic suggests letting $u = \ln x$, hence $dv = x^2\, dx$. We then get $du = (1/x)\, dx$ and $v = x^3/3$ as shown below.

$$\begin{array}{ll} u = \ln x & v = ? \\ du = ? & dv = x^2\ dx \end{array} \quad \Rightarrow \quad \begin{array}{ll} u = \ln x & v = x^3/3 \\ du = 1/x\ dx & dv = x^2\ dx \end{array}$$

Figure 2.10: Setting up Integration by Parts.

The Integration by Parts formula then gives

$$\begin{aligned} \int_1^2 x^2 \ln x\, dx &= \frac{x^3}{3} \ln x \bigg|_1^2 - \int_1^2 \frac{x^3}{3} \frac{1}{x}\, dx \\ &= \frac{x^3}{3} \ln x \bigg|_1^2 - \int_1^2 \frac{x^2}{3}\, dx \\ &= \frac{x^3}{3} \ln x \bigg|_1^2 - \frac{x^3}{9} \bigg|_1^2 \\ &= \left(\frac{x^3}{3} \ln x - \frac{x^3}{9}\right) \bigg|_1^2 \\ &= \left(\frac{8}{3} \ln 2 - \frac{8}{9}\right) - \left(\frac{1}{3} \ln 1 - \frac{1}{9}\right) \\ &= \frac{8}{3} \ln 2 - \frac{7}{9} \\ &\approx 1.07. \end{aligned}$$

In general, Integration by Parts is useful for integrating certain products of functions, like $\int xe^x\, dx$ or $\int x^3 \sin x\, dx$. It is also useful for integrals involving logarithms and inverse trigonometric functions.

As stated before, integration is generally more difficult than derivation. We are developing tools for handling a large array of integrals, and experience will tell us when one tool is preferable/necessary over another.

Notes:

For instance, consider the three similar–looking integrals

$$\int xe^x\,dx, \qquad \int xe^{x^2}\,dx \qquad \text{and} \qquad \int xe^{x^3}\,dx.$$

While the first is calculated easily with Integration by Parts, the second is best approached with Substitution. Taking things one step further, the third integral has no answer in terms of elementary functions, so none of the methods we learn in calculus will get us the exact answer.

Integration by Parts is a very useful method, second only to substitution. In the following sections of this chapter, we continue to learn other integration techniques. The next section focuses on handling integrals containing trigonometric functions.

Notes:

Exercises 2.2

Terms and Concepts

1. T/F: Integration by Parts is useful in evaluating integrands that contain products of functions.

2. T/F: Integration by Parts can be thought of as the "opposite of the Chain Rule."

3. For what is "LIATE" useful?

Problems

In Exercises 4 – 33, evaluate the given indefinite integral.

4. $\int x \sin x \; dx$

5. $\int xe^{-x} \; dx$

6. $\int x^2 \sin x \; dx$

7. $\int x^3 \sin x \; dx$

8. $\int xe^{x^2} \; dx$

9. $\int x^3 e^x \; dx$

10. $\int xe^{-2x} \; dx$

11. $\int e^x \sin x \; dx$

12. $\int e^{2x} \cos x \; dx$

13. $\int e^{2x} \sin(3x) \; dx$

14. $\int e^{5x} \cos(5x) \; dx$

15. $\int \sin x \cos x \; dx$

16. $\int \sin^{-1} x \; dx$

17. $\int \tan^{-1}(2x) \; dx$

18. $\int x \tan^{-1} x \; dx$

19. $\int \sin^{-1} x \; dx$

20. $\int x \ln x \; dx$

21. $\int (x-2) \ln x \; dx$

22. $\int x \ln(x-1) \; dx$

23. $\int x \ln(x^2) \; dx$

24. $\int x^2 \ln x \; dx$

25. $\int (\ln x)^2 \; dx$

26. $\int (\ln(x+1))^2 \; dx$

27. $\int x \sec^2 x \; dx$

28. $\int x \csc^2 x \; dx$

29. $\int x\sqrt{x-2} \; dx$

30. $\int x\sqrt{x^2-2} \; dx$

31. $\int \sec x \tan x \; dx$

32. $\int x \sec x \tan x \; dx$

33. $\int x \csc x \cot x \; dx$

In Exercises 34 – 38, evaluate the indefinite integral after first making a substitution.

34. $\int \sin(\ln x) \; dx$

35. $\int \sin(\sqrt{x}) \; dx$

36. $\int \ln(\sqrt{x}) \; dx$

37. $\int e^{\sqrt{x}} \; dx$

38. $\int e^{\ln x}\ dx$

In Exercises 39 – 47, evaluate the definite integral. Note: the corresponding indefinite integrals appear in Exercises 4 – 12.

39. $\int_0^{\pi} x\sin x\ dx$

40. $\int_{-1}^{1} xe^{-x}\ dx$

41. $\int_{-\pi/4}^{\pi/4} x^2\sin x\ dx$

42. $\int_{-\pi/2}^{\pi/2} x^3\sin x\ dx$

43. $\int_0^{\sqrt{\ln 2}} xe^{x^2}\ dx$

44. $\int_0^{1} x^3e^{x}\ dx$

45. $\int_1^{2} xe^{-2x}\ dx$

46. $\int_0^{\pi} e^{x}\sin x\ dx$

47. $\int_{-\pi/2}^{\pi/2} e^{2x}\cos x\ dx$

2.3 Trigonometric Integrals

This section describes several techniques for finding antiderivatives of certain combinations of trigonometric functions.

Integrals of the form $\int \sin^m x \cos^n x \; dx$

In learning the technique of Substitution, we saw the integral $\int \sin x \cos x \; dx$ in Example 2.1.4. The integration was not difficult, and one could easily evaluate the indefinite integral by letting $u = \sin x$ or by letting $u = \cos x$. This integral is easy since the power of both sine and cosine is 1.

We generalize this integral and consider integrals of the form $\int \sin^m x \cos^n x \; dx$, where m, n are nonnegative integers. Our strategy for evaluating these integrals is to use the identity $\cos^2 x + \sin^2 x = 1$ to convert high powers of one trigonometric function into the other, leaving a single sine or cosine term in the integrand. We summarize the general technique in the following Key Idea.

Key Idea 2.3.1 Integrals Involving Powers of Sine and Cosine

Consider $\int \sin^m x \cos^n x \; dx$, where m, n are nonnegative integers.

1. If m is odd, then $m = 2k+1$ for some integer k. Rewrite

$$\sin^m x = \sin^{2k+1} x = \sin^{2k} x \sin x = (\sin^2 x)^k \sin x = (1-\cos^2 x)^k \sin x.$$

   Then

$$\int \sin^m x \cos^n x \; dx = \int (1-\cos^2 x)^k \sin x \cos^n x \; dx = -\int (1-u^2)^k u^n \; du,$$

   where $u = \cos x$ and $du = -\sin x \; dx$.

2. If n is odd, then using substitutions similar to that outlined above we have

$$\int \sin^m x \cos^n x \; dx = \int u^m (1-u^2)^k \; du,$$

   where $u = \sin x$ and $du = \cos x \; dx$.

3. If both m and n are even, use the power–reducing identities

$$\cos^2 x = \frac{1+\cos(2x)}{2} \quad \text{and} \quad \sin^2 x = \frac{1-\cos(2x)}{2}$$

   to reduce the degree of the integrand. Expand the result and apply the principles of this Key Idea again.

Notes:

We practice applying Key Idea 2.3.1 in the next examples.

Example 2.3.1 Integrating powers of sine and cosine
Evaluate $\int \sin^5 x \cos^8 x\ dx$.

Solution The power of the sine term is odd, so we rewrite $\sin^5 x$ as

$$\sin^5 x = \sin^4 x \sin x = (\sin^2 x)^2 \sin x = (1 - \cos^2 x)^2 \sin x.$$

Our integral is now $\int (1-\cos^2 x)^2 \cos^8 x \sin x\ dx$. Let $u = \cos x$, hence $du = -\sin x\ dx$. Making the substitution and expanding the integrand gives

$$\int (1-\cos^2)^2 \cos^8 x \sin x\ dx = -\int (1-u^2)^2 u^8\ du = -\int \left(1-2u^2+u^4\right) u^8\ du = -\int \left(u^8-2u^{10}+u^{12}\right)\ du.$$

This final integral is not difficult to evaluate, giving

$$\begin{aligned} -\int \left(u^8 - 2u^{10} + u^{12}\right)\ du &= -\frac{1}{9}u^9 + \frac{2}{11}u^{11} - \frac{1}{13}u^{13} + C \\ &= -\frac{1}{9}\cos^9 x + \frac{2}{11}\cos^{11} x - \frac{1}{13}\cos^{13} x + C. \end{aligned}$$

Example 2.3.2 Integrating powers of sine and cosine
Evaluate $\int \sin^5 x \cos^9 x\ dx$.

Solution The powers of both the sine and cosine terms are odd, therefore we can apply the techniques of Key Idea 2.3.1 to either power. We choose to work with the power of the cosine term since the previous example used the sine term's power.

We rewrite $\cos^9 x$ as

$$\begin{aligned} \cos^9 x &= \cos^8 x \cos x \\ &= (\cos^2 x)^4 \cos x \\ &= (1 - \sin^2 x)^4 \cos x \\ &= (1 - 4\sin^2 x + 6\sin^4 x - 4\sin^6 x + \sin^8 x) \cos x. \end{aligned}$$

We rewrite the integral as

$$\int \sin^5 x \cos^9 x\ dx = \int \sin^5 x \left(1-4\sin^2 x+6\sin^4 x-4\sin^6 x+\sin^8 x\right) \cos x\ dx.$$

Notes:

Now substitute and integrate, using $u = \sin x$ and $du = \cos x\ dx$.

$$\int \sin^5 x\big(1 - 4\sin^2 x + 6\sin^4 x - 4\sin^6 x + \sin^8 x\big)\cos x\ dx =$$

$$\begin{aligned}\int u^5(1 - 4u^2 + 6u^4 - 4u^6 + u^8)\ du = \int \left(u^5 - 4u^7 + 6u^9 - 4u^{11} + u^{13}\right)\ du \\ = \frac{1}{6}u^6 - \frac{1}{2}u^8 + \frac{3}{5}u^{10} - \frac{1}{3}u^{12} + \frac{1}{14}u^{14} + C \\ = \frac{1}{6}\sin^6 x - \frac{1}{2}\sin^8 x + \frac{3}{5}\sin^{10} x + \ldots \\ -\frac{1}{3}\sin^{12} x + \frac{1}{14}\sin^{14} x + C.\end{aligned}$$

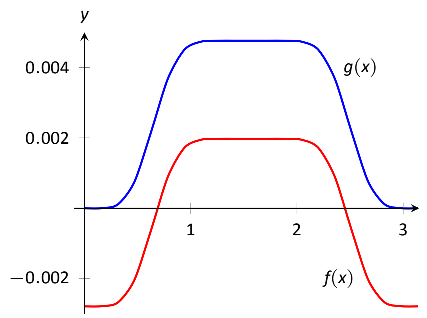


Figure 2.11: A plot of $f(x)$ and $g(x)$ from Example 2.3.2 and the Technology Note.

Technology Note: The work we are doing here can be a bit tedious, but the skills developed (problem solving, algebraic manipulation, etc.) are important. Nowadays problems of this sort are often solved using a computer algebra system. The powerful program *Mathematica*® integrates $\int \sin^5 x \cos^9 x\ dx$ as

$$f(x) = -\frac{45\cos(2x)}{16384} - \frac{5\cos(4x)}{8192} + \frac{19\cos(6x)}{49152} + \frac{\cos(8x)}{4096} - \frac{\cos(10x)}{81920} - \frac{\cos(12x)}{24576} - \frac{\cos(14x)}{114688},$$

which clearly has a different form than our answer in Example 2.3.2, which is

$$g(x) = \frac{1}{6}\sin^6 x - \frac{1}{2}\sin^8 x + \frac{3}{5}\sin^{10} x - \frac{1}{3}\sin^{12} x + \frac{1}{14}\sin^{14} x.$$

Figure 2.11 shows a graph of f and g; they are clearly not equal, but they differ *only by a constant*. That is $g(x) = f(x) + C$ for some constant C. So we have two different antiderivatives of the same function, meaning both answers are correct.

Example 2.3.3 Integrating powers of sine and cosine

Evaluate $\displaystyle\int \cos^4 x \sin^2 x\ dx$.

Solution The powers of sine and cosine are both even, so we employ the power–reducing formulas and algebra as follows.

$$\begin{aligned}\int \cos^4 x\sin^2 x\ dx &= \int \left(\frac{1+\cos(2x)}{2}\right)^2\left(\frac{1-\cos(2x)}{2}\right)\ dx \\ &= \int \frac{1 + 2\cos(2x) + \cos^2(2x)}{4}\cdot\frac{1-\cos(2x)}{2}\ dx \\ &= \int \frac{1}{8}\big(1 + \cos(2x) - \cos^2(2x) - \cos^3(2x)\big)\ dx\end{aligned}$$

Notes:

The $\cos(2x)$ term is easy to integrate, especially with Key Idea 2.1.1. The $\cos^2(2x)$ term is another trigonometric integral with an even power, requiring the power–reducing formula again. The $\cos^3(2x)$ term is a cosine function with an odd power, requiring a substitution as done before. We integrate each in turn below.

$$\int \cos(2x)\ dx = \frac{1}{2}\sin(2x) + C.$$

$$\int \cos^2(2x)\ dx = \int \frac{1+\cos(4x)}{2}\ dx = \frac{1}{2}\Big(x + \frac{1}{4}\sin(4x)\Big) + C.$$

Finally, we rewrite $\cos^3(2x)$ as

$$\cos^3(2x) = \cos^2(2x)\cos(2x) = \big(1-\sin^2(2x)\big)\cos(2x).$$

Letting $u = \sin(2x)$, we have $du = 2\cos(2x)\ dx$, hence

$$\begin{aligned}\int \cos^3(2x)\ dx &= \int \big(1-\sin^2(2x)\big)\cos(2x)\ dx\\ &= \int \frac{1}{2}(1-u^2)\ du\\ &= \frac{1}{2}\Big(u - \frac{1}{3}u^3\Big) + C\\ &= \frac{1}{2}\Big(\sin(2x) - \frac{1}{3}\sin^3(2x)\Big) + C\end{aligned}$$

Putting all the pieces together, we have

$$\begin{aligned}\int \cos^4 x \sin^2 x\ dx &= \int \frac{1}{8}\big(1+\cos(2x)-\cos^2(2x)-\cos^3(2x)\big)\ dx\\ &= \frac{1}{8}\Big[x + \frac{1}{2}\sin(2x) - \frac{1}{2}\big(x+\frac{1}{4}\sin(4x)\big) - \frac{1}{2}\Big(\sin(2x) - \frac{1}{3}\sin^3(2x)\Big)\Big] + C\\ &= \frac{1}{8}\Big[\frac{1}{2}x - \frac{1}{8}\sin(4x) + \frac{1}{6}\sin^3(2x)\Big] + C\end{aligned}$$

The process above was a bit long and tedious, but being able to work a problem such as this from start to finish is important.

Integrals of the form $\int \sin(mx)\sin(nx)\ dx$, $\int \cos(mx)\cos(nx)\ dx$, and $\int \sin(mx)\cos(nx)\ dx$.

Notes:

Functions that contain products of sines and cosines of differing periods are important in many applications including the analysis of sound waves. Integrals of the form

$$\int \sin(mx)\sin(nx)\,dx, \quad \int \cos(mx)\cos(nx)\,dx \quad \text{and} \quad \int \sin(mx)\cos(nx)\,dx$$

are best approached by first applying the Product to Sum Formulas found in the back cover of this text, namely

$$\begin{aligned} \sin(mx)\sin(nx) &= \frac{1}{2}\Big[\cos\big((m-n)x\big) - \cos\big((m+n)x\big)\Big] \\ \cos(mx)\cos(nx) &= \frac{1}{2}\Big[\cos\big((m-n)x\big) + \cos\big((m+n)x\big)\Big] \\ \sin(mx)\cos(nx) &= \frac{1}{2}\Big[\sin\big((m-n)x\big) + \sin\big((m+n)x\big)\Big] \end{aligned}$$

Example 2.3.4 Integrating products of $\sin(mx)$ **and** $\cos(nx)$

Evaluate $\displaystyle\int \sin(5x)\cos(2x)\;dx$.

Solution The application of the formula and subsequent integration are straightforward:

$$\begin{aligned} \int \sin(5x)\cos(2x)\;dx &= \int \frac{1}{2}\Big[\sin(3x)+\sin(7x)\Big]\;dx \\ &= -\frac{1}{6}\cos(3x) - \frac{1}{14}\cos(7x) + C \end{aligned}$$

Integrals of the form $\displaystyle\int \tan^m x \sec^n x\;dx$.

When evaluating integrals of the form $\int \sin^m x \cos^n x\,dx$, the Pythagorean Theorem allowed us to convert even powers of sine into even powers of cosine, and vise–versa. If, for instance, the power of sine was odd, we pulled out one $\sin x$ and converted the remaining even power of $\sin x$ into a function using powers of $\cos x$, leading to an easy substitution.

The same basic strategy applies to integrals of the form $\int \tan^m x \sec^n x\,dx$, albeit a bit more nuanced. The following three facts will prove useful:

- $\frac{d}{dx}(\tan x) = \sec^2 x$,
- $\frac{d}{dx}(\sec x) = \sec x \tan x$, and
- $1 + \tan^2 x = \sec^2 x$ (the Pythagorean Theorem).

Notes:

If the integrand can be manipulated to separate a $\sec^2 x$ term with the remaining secant power even, or if a $\sec x \tan x$ term can be separated with the remaining $\tan x$ power even, the Pythagorean Theorem can be employed, leading to a simple substitution. This strategy is outlined in the following Key Idea.

Key Idea 2.3.2 Integrals Involving Powers of Tangent and Secant

Consider $\displaystyle\int \tan^m x \sec^n x\ dx$, where m, n are nonnegative integers.

1. If n is even, then $n = 2k$ for some integer k. Rewrite $\sec^n x$ as $\sec^n x = \sec^{2k} x = \sec^{2k-2} x \sec^2 x = (1 + \tan^2 x)^{k-1} \sec^2 x$. Then $\displaystyle\int \tan^m x \sec^n x\ dx = \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x\ dx = \int u^m (1 + u^2)^{k-1}\ du$, where $u = \tan x$ and $du = \sec^2 x\ dx$.

2. If m is odd, then $m = 2k + 1$ for some integer k. Rewrite $\tan^m x \sec^n x$ as $\tan^m x \sec^n x = \tan^{2k+1} x \sec^n x = \tan^{2k} x \sec^{n-1} x \sec x \tan x = (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x$. Then $\displaystyle\int \tan^m x \sec^n x\ dx = \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x\ dx = \int (u^2 - 1)^k u^{n-1}\ du$, where $u = \sec x$ and $du = \sec x \tan x\ dx$.

3. If n is odd and m is even, then $m = 2k$ for some integer k. Convert $\tan^m x$ to $(\sec^2 x - 1)^k$. Expand the new integrand and use Integration By Parts, with $dv = \sec^2 x\ dx$.

4. If m is even and $n = 0$, rewrite $\tan^m x$ as $\tan^m x = \tan^{m-2} x \tan^2 x = \tan^{m-2} x (\sec^2 x - 1) = \tan^{m-2} \sec^2 x - \tan^{m-2} x$. So $\displaystyle\int \tan^m x\ dx = \underbrace{\int \tan^{m-2} \sec^2 x\ dx}_{\text{apply rule \#1}} - \underbrace{\int \tan^{m-2} x\ dx}_{\text{apply rule \#4 again}}$.

The techniques described in items 1 and 2 of Key Idea 2.3.2 are relatively straightforward, but the techniques in items 3 and 4 can be rather tedious. A few examples will help with these methods.

Notes:

Example 2.3.5 Integrating powers of tangent and secant

Evaluate $\int \tan^2 x \sec^6 x \; dx$.

Solution Since the power of secant is even, we use rule #1 from Key Idea 2.3.2 and pull out a $\sec^2 x$ in the integrand. We convert the remaining powers of secant into powers of tangent.

$$\begin{aligned}\int \tan^2 x \sec^6 x \; dx &= \int \tan^2 x \sec^4 x \sec^2 x \; dx \\ &= \int \tan^2 x \big(1+\tan^2 x\big)^2 \sec^2 x \; dx\end{aligned}$$

Now substitute, with $u = \tan x$, with $du = \sec^2 x \; dx$.

$$= \int u^2\big(1+u^2\big)^2 \; du$$

We leave the integration and subsequent substitution to the reader. The final answer is

$$= \frac{1}{3}\tan^3 x + \frac{2}{5}\tan^5 x + \frac{1}{7}\tan^7 x + C.$$

Example 2.3.6 Integrating powers of tangent and secant

Evaluate $\int \sec^3 x \; dx$.

Solution We apply rule #3 from Key Idea 2.3.2 as the power of secant is odd and the power of tangent is even (0 is an even number). We use Integration by Parts; the rule suggests letting $dv = \sec^2 x \; dx$, meaning that $u = \sec x$.

$$\begin{array}{ll} u = \sec x & v = \;? \\ du = \;? & dv = \sec^2 x \; dx \end{array} \quad \Rightarrow \quad \begin{array}{ll} u = \sec x & v = \tan x \\ du = \sec x \tan x \; dx & dv = \sec^2 x \; dx \end{array}$$

Figure 2.12: Setting up Integration by Parts.

Employing Integration by Parts, we have

$$\begin{aligned}\int \sec^3 x \; dx &= \int \underbrace{\sec x}_{u} \cdot \underbrace{\sec^2 x \; dx}_{dv} \\ &= \sec x \tan x - \int \sec x \tan^2 x \; dx.\end{aligned}$$

Notes:

This new integral also requires applying rule #3 of Key Idea 2.3.2:

$$\begin{aligned}
&= \sec x \tan x - \int \sec x \big(\sec^2 x - 1 \big) \ dx \\
&= \sec x \tan x - \int \sec^3 x \ dx + \int \sec x \ dx \\
&= \sec x \tan x - \int \sec^3 x \ dx + \ln|\sec x + \tan x|
\end{aligned}$$

In previous applications of Integration by Parts, we have seen where the original integral has reappeared in our work. We resolve this by adding $\int \sec^3 x \ dx$ to both sides, giving:

$$\begin{aligned}
2 \int \sec^3 x \ dx &= \sec x \tan x + \ln|\sec x + \tan x| \\
\int \sec^3 x \ dx &= \frac{1}{2}\Big(\sec x \tan x + \ln|\sec x + \tan x| \Big) + C
\end{aligned}$$

We give one more example.

Example 2.3.7 Integrating powers of tangent and secant

Evaluate $\displaystyle\int \tan^6 x \ dx$.

Solution We employ rule #4 of Key Idea 2.3.2.

$$\begin{aligned}
\int \tan^6 x \ dx &= \int \tan^4 x \tan^2 x \ dx \\
&= \int \tan^4 x \big(\sec^2 x - 1 \big) \ dx \\
&= \int \tan^4 x \sec^2 x \ dx - \int \tan^4 x \ dx
\end{aligned}$$

Integrate the first integral with substitution, $u = \tan x$; integrate the second by employing rule #4 again.

$$\begin{aligned}
&= \frac{1}{5}\tan^5 x - \int \tan^2 x \tan^2 x \ dx \\
&= \frac{1}{5}\tan^5 x - \int \tan^2 x \big(\sec^2 x - 1 \big) \ dx \\
&= \frac{1}{5}\tan^5 x - \int \tan^2 x \sec^2 x \ dx + \int \tan^2 x \ dx
\end{aligned}$$

Notes:

Again, use substitution for the first integral and rule #4 for the second.

$$= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \int \left(\sec^2 x - 1\right)\,dx$$
$$= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + C.$$

These latter examples were admittedly long, with repeated applications of the same rule. Try to not be overwhelmed by the length of the problem, but rather admire how robust this solution method is. A trigonometric function of a high power can be systematically reduced to trigonometric functions of lower powers until all antiderivatives can be computed.

The next section introduces an integration technique known as Trigonometric Substitution, a clever combination of Substitution and the Pythagorean Theorem.

Notes:

Exercises 2.3

Terms and Concepts

1. T/F: $\int \sin^2 x \cos^2 x \, dx$ cannot be evaluated using the techniques described in this section since both powers of $\sin x$ and $\cos x$ are even.

2. T/F: $\int \sin^3 x \cos^3 x \, dx$ cannot be evaluated using the techniques described in this section since both powers of $\sin x$ and $\cos x$ are odd.

3. T/F: This section addresses how to evaluate indefinite integrals such as $\int \sin^5 x \tan^3 x \, dx$.

Problems

In Exercises 4 – 26, evaluate the indefinite integral.

4. $\int \sin x \cos^4 x \, dx$

5. $\int \sin^3 x \cos x \, dx$

6. $\int \sin^3 x \cos^2 x \, dx$

7. $\int \sin^3 x \cos^3 x \, dx$

8. $\int \sin^6 x \cos^5 x \, dx$

9. $\int \sin^2 x \cos^7 x \, dx$

10. $\int \sin^2 x \cos^2 x \, dx$

11. $\int \sin(5x) \cos(3x) \, dx$

12. $\int \sin(x) \cos(2x) \, dx$

13. $\int \sin(3x) \sin(7x) \, dx$

14. $\int \sin(\pi x) \sin(2\pi x) \, dx$

15. $\int \cos(x) \cos(2x) \, dx$

16. $\int \cos\left(\frac{\pi}{2}x\right) \cos(\pi x) \, dx$

17. $\int \tan^4 x \sec^2 x \, dx$

18. $\int \tan^2 x \sec^4 x \, dx$

19. $\int \tan^3 x \sec^4 x \, dx$

20. $\int \tan^3 x \sec^2 x \, dx$

21. $\int \tan^3 x \sec^3 x \, dx$

22. $\int \tan^5 x \sec^5 x \, dx$

23. $\int \tan^4 x \, dx$

24. $\int \sec^5 x \, dx$

25. $\int \tan^2 x \sec x \, dx$

26. $\int \tan^2 x \sec^3 x \, dx$

In Exercises 27 – 33, evaluate the definite integral. Note: the corresponding indefinite integrals appear in the previous set.

27. $\int_0^{\pi} \sin x \cos^4 x \, dx$

28. $\int_{-\pi}^{\pi} \sin^3 x \cos x \, dx$

29. $\int_{-\pi/2}^{\pi/2} \sin^2 x \cos^7 x \, dx$

30. $\int_0^{\pi/2} \sin(5x) \cos(3x) \, dx$

31. $\int_{-\pi/2}^{\pi/2} \cos(x) \cos(2x) \, dx$

32. $\int_0^{\pi/4} \tan^4 x \sec^2 x \, dx$

33. $\int_{-\pi/4}^{\pi/4} \tan^2 x \sec^4 x \, dx$

2.4 Trigonometric Substitution

We have learned a number of integration techniques, including Substitution and Integration by Parts, yet we are still unable to evaluate the above integral without resorting to a geometric interpretation. This section introduces Trigonometric Substitution, a method of integration that fills this gap in our integration skill. This technique works on the same principle as Substitution as found in Section 2.1, though it can feel "backward."

In Section 2.1, we set $u = f(x)$, for some function f, and replaced $f(x)$ with u. In this section, we will set $x = f(\theta)$, where f is a trigonometric function, then replace x with $f(\theta)$.

We start by demonstrating this method in evaluating the following integral. After the example, we will generalize the method and give more examples.

Example 2.4.1 Using Trigonometric Substitution

Evaluate $\displaystyle\int_{-3}^{3} \sqrt{9-x^2}\ dx$.

Solution We begin by noting that $9\sin^2\theta + 9\cos^2\theta = 9$, and hence $9\cos^2\theta = 9 - 9\sin^2\theta$. If we let $x = 3\sin\theta$, then $9 - x^2 = 9 - 9\sin^2\theta = 9\cos^2\theta$.

Setting $x = 3\sin\theta$ gives $dx = 3\cos\theta\ d\theta$. We are almost ready to substitute. We also wish to change our bounds of integration. The bound $x = -3$ corresponds to $\theta = -\pi/2$ (for when $\theta = -\pi/2$, $x = 3\sin\theta = -3$). Likewise, the bound of $x = 3$ is replaced by the bound $\theta = \pi/2$. Thus

$$\begin{aligned}\int_{-3}^{3} \sqrt{9-x^2}\ dx &= \int_{-\pi/2}^{\pi/2} \sqrt{9-9\sin^2\theta}(3\cos\theta)\ d\theta \\ &= \int_{-\pi/2}^{\pi/2} 3\sqrt{9\cos^2\theta}\cos\theta\ d\theta \\ &= \int_{-\pi/2}^{\pi/2} 3|3\cos\theta|\cos\theta\ d\theta.\end{aligned}$$

On $[-\pi/2, \pi/2]$, $\cos\theta$ is always positive, so we can drop the absolute value bars, then employ a power–reducing formula:

Notes:

$$= \int_{-\pi/2}^{\pi/2} 9\cos^2\theta \; d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{9}{2}\big(1+\cos(2\theta)\big) \; d\theta$$

$$= \frac{9}{2}\big(\theta + \frac{1}{2}\sin(2\theta)\big)\Big|_{-\pi/2}^{\pi/2} = \frac{9}{2}\pi.$$

This matches our answer from before.

We now describe in detail Trigonometric Substitution. This method excels when dealing with integrands that contain $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$ and $\sqrt{x^2+a^2}$. The following Key Idea outlines the procedure for each case, followed by more examples. Each right triangle acts as a reference to help us understand the relationships between x and θ.

Key Idea 2.4.1 Trigonometric Substitution

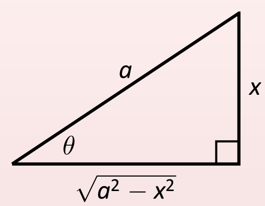
(a) For integrands containing $\sqrt{a^2-x^2}$:

Let $x = a\sin\theta$, $\qquad dx = a\cos\theta\; d\theta$

Thus $\theta = \sin^{-1}(x/a)$, for $-\pi/2 \le \theta \le \pi/2$.

On this interval, $\cos\theta \ge 0$, so

$\sqrt{a^2-x^2} = a\cos\theta$



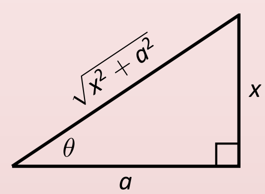
(b) For integrands containing $\sqrt{x^2+a^2}$:

Let $x = a\tan\theta$, $\qquad dx = a\sec^2\theta\; d\theta$

Thus $\theta = \tan^{-1}(x/a)$, for $-\pi/2 < \theta < \pi/2$.

On this interval, $\sec\theta > 0$, so

$\sqrt{x^2+a^2} = a\sec\theta$

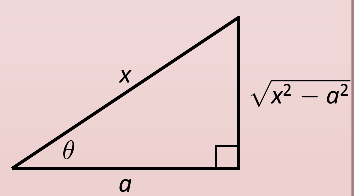


(c) For integrands containing $\sqrt{x^2-a^2}$:

Let $x = a\sec\theta$, $\qquad dx = a\sec\theta\tan\theta\; d\theta$

Thus $\theta = \sec^{-1}(x/a)$. If $x/a \ge 1$, then $0 \le \theta < \pi/2$; if $x/a \le -1$, then $\pi/2 < \theta \le \pi$.

We restrict our work to where $x \ge a$, so $x/a \ge 1$, and $0 \le \theta < \pi/2$. On this interval, $\tan\theta \ge 0$, so $\sqrt{x^2-a^2} = a\tan\theta$



Notes:

Example 2.4.2 Using Trigonometric Substitution

Evaluate $\displaystyle\int \frac{1}{\sqrt{5+x^2}}\,dx$.

Solution Using Key Idea 2.4.1(b), we recognize $a=\sqrt{5}$ and set $x=\sqrt{5}\tan\theta$. This makes $dx=\sqrt{5}\sec^2\theta\ d\theta$. We will use the fact that $\sqrt{5+x^2}=\sqrt{5+5\tan^2\theta}=\sqrt{5\sec^2\theta}=\sqrt{5}\sec\theta$. Substituting, we have:

$$\begin{aligned}\int \frac{1}{\sqrt{5+x^2}}\,dx &= \int \frac{1}{\sqrt{5+5\tan^2\theta}}\sqrt{5}\sec^2\theta\ d\theta\\ &= \int \frac{\sqrt{5}\sec^2\theta}{\sqrt{5}\sec\theta}\ d\theta\\ &= \int \sec\theta\ d\theta\\ &= \ln\big|\sec\theta+\tan\theta\big|+C.\end{aligned}$$

While the integration steps are over, we are not yet done. The original problem was stated in terms of x, whereas our answer is given in terms of θ. We must convert back to x.

The reference triangle given in Key Idea 2.4.1(b) helps. With $x=\sqrt{5}\tan\theta$, we have

$$\tan\theta=\frac{x}{\sqrt{5}}\quad\text{and}\quad\sec\theta=\frac{\sqrt{x^2+5}}{\sqrt{5}}.$$

This gives

$$\begin{aligned}\int \frac{1}{\sqrt{5+x^2}}\,dx &= \ln\big|\sec\theta+\tan\theta\big|+C\\ &= \ln\left|\frac{\sqrt{x^2+5}}{\sqrt{5}}+\frac{x}{\sqrt{5}}\right|+C.\end{aligned}$$

We can leave this answer as is, or we can use a logarithmic identity to simplify it. Note:

$$\begin{aligned}\ln\left|\frac{\sqrt{x^2+5}}{\sqrt{5}}+\frac{x}{\sqrt{5}}\right|+C &= \ln\left|\frac{1}{\sqrt{5}}\left(\sqrt{x^2+5}+x\right)\right|+C\\ &= \ln\left|\frac{1}{\sqrt{5}}\right|+\ln\left|\sqrt{x^2+5}+x\right|+C\\ &= \ln\left|\sqrt{x^2+5}+x\right|+C,\end{aligned}$$

Notes:

where the $\ln\left(1/\sqrt{5}\right)$ term is absorbed into the constant C. (In Section **??** we will learn another way of approaching this problem.)

Example 2.4.3 Using Trigonometric Substitution

Evaluate $\displaystyle\int \sqrt{4x^2-1}\ dx$.

Solution We start by rewriting the integrand so that it looks like $\sqrt{x^2-a^2}$ for some value of a:

$$\begin{aligned}\sqrt{4x^2-1} &= \sqrt{4\left(x^2-\frac{1}{4}\right)}\\ &= 2\sqrt{x^2-\left(\frac{1}{2}\right)^2}.\end{aligned}$$

So we have $a=1/2$, and following Key Idea 2.4.1(c), we set $x=\frac{1}{2}\sec\theta$, and hence $dx=\frac{1}{2}\sec\theta\tan\theta\ d\theta$. We now rewrite the integral with these substitutions:

$$\begin{aligned}\int\sqrt{4x^2-1}\ dx &= \int 2\sqrt{x^2-\left(\frac{1}{2}\right)^2}\ dx\\ &= \int 2\sqrt{\frac{1}{4}\sec^2\theta-\frac{1}{4}}\left(\frac{1}{2}\sec\theta\tan\theta\right)\ d\theta\\ &= \int\sqrt{\frac{1}{4}(\sec^2\theta-1)}\Big(\sec\theta\tan\theta\Big)\ d\theta\\ &= \int\sqrt{\frac{1}{4}\tan^2\theta}\Big(\sec\theta\tan\theta\Big)\ d\theta\\ &= \int\frac{1}{2}\tan^2\theta\sec\theta\ d\theta\\ &= \frac{1}{2}\int\Big(\sec^2\theta-1\Big)\sec\theta\ d\theta\\ &= \frac{1}{2}\int\big(\sec^3\theta-\sec\theta\big)\ d\theta.\end{aligned}$$

We integrated $\sec^3\theta$ in Example 2.3.6, finding its antiderivatives to be

$$\int\sec^3\theta\ d\theta = \frac{1}{2}\Big(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\Big)+C.$$

Notes:

Thus

$$\begin{aligned}\int \sqrt{4x^2-1}\,dx &= \frac{1}{2}\int \big(\sec^3\theta - \sec\theta\big)\,d\theta\\ &= \frac{1}{2}\left(\frac{1}{2}\Big(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\Big) - \ln|\sec\theta+\tan\theta|\right) + C\\ &= \frac{1}{4}\left(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\right) + C.\end{aligned}$$

We are not yet done. Our original integral is given in terms of x, whereas our final answer, as given, is in terms of θ. We need to rewrite our answer in terms of x. With $a = 1/2$, and $x = \frac{1}{2}\sec\theta$, the reference triangle in Key Idea 2.4.1(c) shows that

$$\tan\theta = \sqrt{x^2-1/4}\Big/(1/2) = 2\sqrt{x^2-1/4} \quad \text{and} \quad \sec\theta = 2x.$$

Thus

$$\begin{aligned}\frac{1}{4}\Big(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\Big) + C &= \frac{1}{4}\Big(2x\cdot 2\sqrt{x^2-1/4} - \ln\big|2x+2\sqrt{x^2-1/4}\big|\Big) + C\\ &= \frac{1}{4}\Big(4x\sqrt{x^2-1/4} - \ln\big|2x+2\sqrt{x^2-1/4}\big|\Big) + C.\end{aligned}$$

The final answer is given in the last line above, repeated here:

$$\int \sqrt{4x^2-1}\,dx = \frac{1}{4}\Big(4x\sqrt{x^2-1/4} - \ln\big|2x+2\sqrt{x^2-1/4}\big|\Big) + C.$$

Example 2.4.4 Using Trigonometric Substitution

Evaluate $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}\,dx$.

Solution We use Key Idea 2.4.1(a) with $a = 2$, $x = 2\sin\theta$, $dx = 2\cos\theta$ and hence $\sqrt{4-x^2} = 2\cos\theta$. This gives

$$\begin{aligned}\int \frac{\sqrt{4-x^2}}{x^2}\,dx &= \int \frac{2\cos\theta}{4\sin^2\theta}(2\cos\theta)\,d\theta\\ &= \int \cot^2\theta\,d\theta\\ &= \int (\csc^2\theta - 1)\,d\theta\\ &= -\cot\theta - \theta + C.\end{aligned}$$

We need to rewrite our answer in terms of x. Using the reference triangle found in Key Idea 2.4.1(a), we have $\cot\theta = \sqrt{4-x^2}/x$ and $\theta =$

Notes:

$\sin^{-1}(x/2)$. Thus

$$\int \frac{\sqrt{4-x^2}}{x^2}\ dx = -\frac{\sqrt{4-x^2}}{x} - \sin^{-1}\left(\frac{x}{2}\right) + C.$$

Trigonometric Substitution can be applied in many situations, even those not of the form $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$ or $\sqrt{x^2+a^2}$. In the following example, we apply it to an integral we already know how to handle.

Example 2.4.5 Using Trigonometric Substitution

Evaluate $\displaystyle\int \frac{1}{x^2+1}\ dx$.

Solution We know the answer already as $\tan^{-1} x + C$. We apply Trigonometric Substitution here to show that we get the same answer without inherently relying on knowledge of the derivative of the arctangent function.

Using Key Idea 2.4.1(b), let $x = \tan\theta$, $dx = \sec^2\theta\ d\theta$ and note that $x^2+1 = \tan^2\theta + 1 = \sec^2\theta$. Thus

$$\begin{aligned}\int \frac{1}{x^2+1}\ dx &= \int \frac{1}{\sec^2\theta}\sec^2\theta\ d\theta\\ &= \int 1\ d\theta\\ &= \theta + C.\end{aligned}$$

Since $x = \tan\theta$, $\theta = \tan^{-1} x$, and we conclude that $\displaystyle\int \frac{1}{x^2+1}\ dx = \tan^{-1} x + C$.

The next example is similar to the previous one in that it does not involve a square–root. It shows how several techniques and identities can be combined to obtain a solution.

Example 2.4.6 Using Trigonometric Substitution

Evaluate $\displaystyle\int \frac{1}{(x^2+6x+10)^2}\ dx$.

Solution We start by completing the square, then make the substitution $u = x+3$, followed by the trigonometric substitution of $u = \tan\theta$:

$$\int \frac{1}{(x^2+6x+10)^2}\ dx = \int \frac{1}{\big((x+3)^2+1\big)^2}\ dx = \int \frac{1}{(u^2+1)^2}\ du.$$

Notes:

Now make the substitution $u = \tan\theta$, $du = \sec^2\theta\ d\theta$:

$$= \int \frac{1}{(\tan^2\theta + 1)^2} \sec^2\theta\ d\theta$$
$$= \int \frac{1}{(\sec^2\theta)^2} \sec^2\theta\ d\theta$$
$$= \int \cos^2\theta\ d\theta.$$

Applying a power reducing formula, we have

$$= \int \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right)\ d\theta$$
$$= \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C. \qquad (2.1)$$

We need to return to the variable x. As $u = \tan\theta$, $\theta = \tan^{-1} u$. Using the identity $\sin(2\theta) = 2\sin\theta\cos\theta$ and using the reference triangle found in Key Idea 2.4.1(b), we have

$$\frac{1}{4}\sin(2\theta) = \frac{1}{2}\frac{u}{\sqrt{u^2+1}} \cdot \frac{1}{\sqrt{u^2+1}} = \frac{1}{2}\frac{u}{u^2+1}.$$

Finally, we return to x with the substitution $u = x + 3$. We start with the expression in Equation (2.1):

$$\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C = \frac{1}{2}\tan^{-1} u + \frac{1}{2}\frac{u}{u^2+1} + C$$
$$= \frac{1}{2}\tan^{-1}(x+3) + \frac{x+3}{2(x^2+6x+10)} + C.$$

Stating our final result in one line,

$$\int \frac{1}{(x^2+6x+10)^2}\ dx = \frac{1}{2}\tan^{-1}(x+3) + \frac{x+3}{2(x^2+6x+10)} + C.$$

Our last example returns us to definite integrals, as seen in our first example. Given a definite integral that can be evaluated using Trigonometric Substitution, we could first evaluate the corresponding indefinite integral (by changing from an integral in terms of x to one in terms of θ, then converting back to x) and then evaluate using the original bounds.

Notes:

It is much more straightforward, though, to change the bounds as we substitute.

Example 2.4.7 Definite integration and Trigonometric Substitution

Evaluate $\displaystyle\int_0^5 \frac{x^2}{\sqrt{x^2+25}}\ dx$.

Solution Using Key Idea 2.4.1(b), we set $x = 5\tan\theta$, $dx = 5\sec^2\theta\ d\theta$, and note that $\sqrt{x^2+25} = 5\sec\theta$. As we substitute, we can also change the bounds of integration.

The lower bound of the original integral is $x = 0$. As $x = 5\tan\theta$, we solve for θ and find $\theta = \tan^{-1}(x/5)$. Thus the new lower bound is $\theta = \tan^{-1}(0) = 0$. The original upper bound is $x = 5$, thus the new upper bound is $\theta = \tan^{-1}(5/5) = \pi/4$.

Thus we have

$$\begin{aligned}\int_0^5 \frac{x^2}{\sqrt{x^2+25}}\ dx &= \int_0^{\pi/4} \frac{25\tan^2\theta}{5\sec\theta} 5\sec^2\theta\ d\theta \\ &= 25\int_0^{\pi/4} \tan^2\theta\sec\theta\ d\theta.\end{aligned}$$

We encountered this indefinite integral in Example 2.4.3 where we found

$$\int \tan^2\theta\sec\theta\ d\theta = \frac{1}{2}\big(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\big).$$

So

$$\begin{aligned}25\int_0^{\pi/4} \tan^2\theta\sec\theta\ d\theta &= \frac{25}{2}\big(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\big)\Big|_0^{\pi/4} \\ &= \frac{25}{2}\big(\sqrt{2} - \ln(\sqrt{2}+1)\big) \\ &\approx 6.661.\end{aligned}$$

The following equalities are very useful when evaluating integrals using Trigonometric Substitution.

Notes:

Key Idea 2.4.2 Useful Equalities with Trigonometric Substitution

1. $\sin(2\theta) = 2\sin\theta\cos\theta$
2. $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
3. $\displaystyle\int \sec^3\theta \; d\theta = \frac{1}{2}\Big(\sec\theta\tan\theta + \ln\big|\sec\theta + \tan\theta\big|\Big) + C$
4. $\displaystyle\int \cos^2\theta \; d\theta = \int \frac{1}{2}\big(1 + \cos(2\theta)\big) \; d\theta = \frac{1}{2}\big(\theta + \sin\theta\cos\theta\big) + C.$

The next section introduces Partial Fraction Decomposition, which is an algebraic technique that turns "complicated" fractions into sums of "simpler" fractions, making integration easier.

Notes:

Exercises 2.4

Terms and Concepts

1. Trigonometric Substitution works on the same principles as Integration by Substitution, though it can feel "________".

2. If one uses Trigonometric Substitution on an integrand containing $\sqrt{25-x^2}$, then one should set $x =$________.

3. Consider the Pythagorean Identity $\sin^2\theta + \cos^2\theta = 1$.
   (a) What identity is obtained when both sides are divided by $\cos^2\theta$?
   (b) Use the new identity to simplify $9\tan^2\theta + 9$.

4. Why does Key Idea 2.4.1(a) state that $\sqrt{a^2-x^2} = a\cos\theta$, and not $|a\cos\theta|$?

Problems

In Exercises 5 – 16, apply Trigonometric Substitution to evaluate the indefinite integrals.

5. $\displaystyle\int \sqrt{x^2+1}\,dx$

6. $\displaystyle\int \sqrt{x^2+4}\,dx$

7. $\displaystyle\int \sqrt{1-x^2}\,dx$

8. $\displaystyle\int \sqrt{9-x^2}\,dx$

9. $\displaystyle\int \sqrt{x^2-1}\,dx$

10. $\displaystyle\int \sqrt{x^2-16}\,dx$

11. $\displaystyle\int \sqrt{4x^2+1}\,dx$

12. $\displaystyle\int \sqrt{1-9x^2}\,dx$

13. $\displaystyle\int \sqrt{16x^2-1}\,dx$

14. $\displaystyle\int \frac{8}{\sqrt{x^2+2}}\,dx$

15. $\displaystyle\int \frac{3}{\sqrt{7-x^2}}\,dx$

16. $\displaystyle\int \frac{5}{\sqrt{x^2-8}}\,dx$

In Exercises 17 – 26, evaluate the indefinite integrals. Some may be evaluated without Trigonometric Substitution.

17. $\displaystyle\int \frac{\sqrt{x^2-11}}{x}\,dx$

18. $\displaystyle\int \frac{1}{(x^2+1)^2}\,dx$

19. $\displaystyle\int \frac{x}{\sqrt{x^2-3}}\,dx$

20. $\displaystyle\int x^2\sqrt{1-x^2}\,dx$

21. $\displaystyle\int \frac{x}{(x^2+9)^{3/2}}\,dx$

22. $\displaystyle\int \frac{5x^2}{\sqrt{x^2-10}}\,dx$

23. $\displaystyle\int \frac{1}{(x^2+4x+13)^2}\,dx$

24. $\displaystyle\int x^2(1-x^2)^{-3/2}\,dx$

25. $\displaystyle\int \frac{\sqrt{5-x^2}}{7x^2}\,dx$

26. $\displaystyle\int \frac{x^2}{\sqrt{x^2+3}}\,dx$

In Exercises 27 – 32, evaluate the definite integrals by making the proper trigonometric substitution and changing the bounds of integration. (Note: each of the corresponding indefinite integrals has appeared previously in this Exercise set.)

27. $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,dx$

28. $\displaystyle\int_{4}^{8} \sqrt{x^2-16}\,dx$

29. $\displaystyle\int_{0}^{2} \sqrt{x^2+4}\,dx$

30. $\displaystyle\int_{-1}^{1} \frac{1}{(x^2+1)^2}\,dx$

31. $\displaystyle\int_{-1}^{1} \sqrt{9-x^2}\,dx$

32. $\displaystyle\int_{-1}^{1} x^2\sqrt{1-x^2}\,dx$

2.5 Partial Fraction Decomposition

In this section we investigate the antiderivatives of rational functions. Recall that rational functions are functions of the form $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$. Such functions arise in many contexts, one of which is the solving of certain fundamental differential equations.

We begin with an example that demonstrates the motivation behind this section. Consider the integral $\displaystyle\int \frac{1}{x^2-1}\,dx$.

We do not have a simple formula for this (if the denominator were x^2+1, we would recognize the antiderivative as being the arctangent function). It can be solved using Trigonometric Substitution, but note how the integral is easy to evaluate once we realize:

$$\frac{1}{x^2-1} = \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

Thus

$$\begin{aligned}\int \frac{1}{x^2-1}\,dx &= \int \frac{1/2}{x-1}\,dx - \int \frac{1/2}{x+1}\,dx \\ &= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.\end{aligned}$$

This section teaches how to *decompose*

$$\frac{1}{x^2-1} \quad \text{into} \quad \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

We start with a rational function $f(x) = \dfrac{p(x)}{q(x)}$, where p and q do not have any common factors and the degree of p is less than the degree of q. It can be shown that any polynomial, and hence q, can be factored into a product of linear and irreducible quadratic terms. The following Key Idea states how to decompose a rational function into a sum of rational functions whose denominators are all of lower degree than q.

Notes:

Key Idea 2.5.1 Partial Fraction Decomposition

Let $\dfrac{p(x)}{q(x)}$ be a rational function, where the degree of p is less than the degree of q.

1. **Linear Terms:** Let $(x-a)$ divide $q(x)$, where $(x-a)^n$ is the highest power of $(x-a)$ that divides $q(x)$. Then the decomposition of $\frac{p(x)}{q(x)}$ will contain the sum

$$\frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}.$$

2. **Quadratic Terms:** Let x^2+bx+c divide $q(x)$, where $(x^2+bx+c)^n$ is the highest power of x^2+bx+c that divides $q(x)$. Then the decomposition of $\frac{p(x)}{q(x)}$ will contain the sum

$$\frac{B_1x+C_1}{x^2+bx+c} + \frac{B_2x+C_2}{(x^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(x^2+bx+c)^n}.$$

To find the coefficients A_i, B_i and C_i:

1. Multiply all fractions by $q(x)$, clearing the denominators. Collect like terms.

2. Equate the resulting coefficients of the powers of x and solve the resulting system of linear equations.

The following examples will demonstrate how to put this Key Idea into practice. Example 2.5.1 stresses the decomposition aspect of the Key Idea.

Example 2.5.1 Decomposing into partial fractions

Decompose $f(x) = \dfrac{1}{(x+5)(x-2)^3(x^2+x+2)(x^2+x+7)^2}$ without solving for the resulting coefficients.

Solution The denominator is already factored, as both x^2+x+2 and x^2+x+7 cannot be factored further. We need to decompose $f(x)$ properly. Since $(x+5)$ is a linear term that divides the denominator, there will be a

$$\frac{A}{x+5}$$

Notes:

term in the decomposition.

As $(x-2)^3$ divides the denominator, we will have the following terms in the decomposition:

$$\frac{B}{x-2}, \quad \frac{C}{(x-2)^2} \quad \text{and} \quad \frac{D}{(x-2)^3}.$$

The x^2+x+2 term in the denominator results in a $\dfrac{Ex+F}{x^2+x+2}$ term.

Finally, the $(x^2+x+7)^2$ term results in the terms

$$\frac{Gx+H}{x^2+x+7} \quad \text{and} \quad \frac{Ix+J}{(x^2+x+7)^2}.$$

All together, we have

$$\frac{1}{(x+5)(x-2)^3(x^2+x+2)(x^2+x+7)^2} = \frac{A}{x+5} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3} + \frac{Ex+F}{x^2+x+2} + \frac{Gx+H}{x^2+x+7} + \frac{Ix+J}{(x^2+x+7)^2}$$

Solving for the coefficients A, $B \ldots J$ would be a bit tedious but not "hard."

Example 2.5.2 Decomposing into partial fractions

Perform the partial fraction decomposition of $\dfrac{1}{x^2-1}$.

Solution The denominator factors into two linear terms: $x^2-1 = (x-1)(x+1)$. Thus

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}.$$

To solve for A and B, first multiply through by $x^2-1 = (x-1)(x+1)$:

$$\begin{aligned} 1 &= \frac{A(x-1)(x+1)}{x-1} + \frac{B(x-1)(x+1)}{x+1} \\ &= A(x+1) + B(x-1) \\ &= Ax + A + Bx - B \end{aligned}$$

Now collect like terms.

$$= (A+B)x + (A-B).$$

Notes:

The next step is key. Note the equality we have:

$$1 = (A+B)x + (A-B).$$

For clarity's sake, rewrite the left hand side as

$$0x + 1 = (A+B)x + (A-B).$$

On the left, the coefficient of the x term is 0; on the right, it is $(A+B)$. Since both sides are equal, we must have that $0 = A+B$.

Likewise, on the left, we have a constant term of 1; on the right, the constant term is $(A-B)$. Therefore we have $1 = A-B$.

We have two linear equations with two unknowns. This one is easy to solve by hand, leading to

$$\begin{array}{c} A+B=0 \\ A-B=1 \end{array} \Rightarrow \begin{array}{c} A=1/2 \\ B=-1/2 \end{array}.$$

Thus

$$\frac{1}{x^2-1} = \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

Example 2.5.3 Integrating using partial fractions

Use partial fraction decomposition to integrate $\displaystyle\int \frac{1}{(x-1)(x+2)^2}\,dx$.

Solution We decompose the integrand as follows, as described by Key Idea 2.5.1:

$$\frac{1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}.$$

To solve for A, B and C, we multiply both sides by $(x-1)(x+2)^2$ and collect like terms:

$$\begin{aligned} 1 &= A(x+2)^2 + B(x-1)(x+2) + C(x-1) \qquad (2.2) \\ &= Ax^2 + 4Ax + 4A + Bx^2 + Bx - 2B + Cx - C \\ &= (A+B)x^2 + (4A+B+C)x + (4A-2B-C) \end{aligned}$$

We have

$$0x^2 + 0x + 1 = (A+B)x^2 + (4A+B+C)x + (4A-2B-C)$$

leading to the equations

$$A+B=0, \quad 4A+B+C=0 \quad \text{and} \quad 4A-2B-C=1.$$

Note: Equation 2.2 offers a direct route to finding the values of A, B and C. Since the equation holds for all values of x, it holds in particular when $x=1$. However, when $x=1$, the right hand side simplifies to $A(1+2)^2 = 9A$. Since the left hand side is still 1, we have $1 = 9A$. Hence $A = 1/9$.
Likewise, the equality holds when $x = -2$; this leads to the equation $1 = -3C$. Thus $C = -1/3$.
Knowing A and C, we can find the value of B by choosing yet another value of x, such as $x = 0$, and solving for B.

Notes:

These three equations of three unknowns lead to a unique solution:

$$A = 1/9, \quad B = -1/9 \quad \text{and} \quad C = -1/3.$$

Thus

$$\int \frac{1}{(x-1)(x+2)^2}\,dx = \int \frac{1/9}{x-1}\,dx + \int \frac{-1/9}{x+2}\,dx + \int \frac{-1/3}{(x+2)^2}\,dx.$$

Each can be integrated with a simple substitution with $u = x - 1$ or $u = x + 2$ (or by directly applying Key Idea 2.1.1 as the denominators are linear functions). The end result is

$$\int \frac{1}{(x-1)(x+2)^2}\,dx = \frac{1}{9}\ln|x-1| - \frac{1}{9}\ln|x+2| + \frac{1}{3(x+2)} + C.$$

Example 2.5.4 Integrating using partial fractions

Use partial fraction decomposition to integrate $\displaystyle\int \frac{x^3}{(x-5)(x+3)}\,dx.$

Note: The values of A and B can be quickly found using the technique described in the margin of Example 2.5.3.

Solution Key Idea 2.5.1 presumes that the degree of the numerator is less than the degree of the denominator. Since this is not the case here, we begin by using polynomial division to reduce the degree of the numerator. We omit the steps, but encourage the reader to verify that

$$\frac{x^3}{(x-5)(x+3)} = x + 2 + \frac{19x+30}{(x-5)(x+3)}.$$

Using Key Idea 2.5.1, we can rewrite the new rational function as:

$$\frac{19x+30}{(x-5)(x+3)} = \frac{A}{x-5} + \frac{B}{x+3}$$

for appropriate values of A and B. Clearing denominators, we have

$$\begin{aligned} 19x + 30 &= A(x+3) + B(x-5) \\ &= (A+B)x + (3A - 5B). \end{aligned}$$

This implies that:

$$\begin{aligned} 19 &= A + B \\ 30 &= 3A - 5B. \end{aligned}$$

Notes:

Solving this system of linear equations gives

$$\begin{aligned}125/8 &= A\\ 27/8 &= B.\end{aligned}$$

We can now integrate.

$$\begin{aligned}\int \frac{x^3}{(x-5)(x+3)}\,dx &= \int \left(x+2+\frac{125/8}{x-5}+\frac{27/8}{x+3}\right)\,dx\\ &= \frac{x^2}{2}+2x+\frac{125}{8}\ln|x-5|+\frac{27}{8}\ln|x+3|+C.\end{aligned}$$

Example 2.5.5 Integrating using partial fractions

Use partial fraction decomposition to evaluate $\displaystyle\int \frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\,dx.$

Solution The degree of the numerator is less than the degree of the denominator so we begin by applying Key Idea 2.5.1. We have:

$$\frac{7x^2+31x+54}{(x+1)(x^2+6x+11)} = \frac{A}{x+1}+\frac{Bx+C}{x^2+6x+11}.$$

Now clear the denominators.

$$\begin{aligned}7x^2+31x+54 &= A(x^2+6x+11)+(Bx+C)(x+1)\\ &= (A+B)x^2+(6A+B+C)x+(11A+C).\end{aligned}$$

This implies that:

$$\begin{aligned}7 &= A+B\\ 31 &= 6A+B+C\\ 54 &= 11A+C.\end{aligned}$$

Solving this system of linear equations gives the nice result of $A=5$, $B=2$ and $C=-1$. Thus

$$\int \frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\,dx = \int\left(\frac{5}{x+1}+\frac{2x-1}{x^2+6x+11}\right)\,dx.$$

The first term of this new integrand is easy to evaluate; it leads to a $5\ln|x+1|$ term. The second term is not hard, but takes several steps and uses substitution techniques.

Notes:

The integrand $\frac{2x-1}{x^2+6x+11}$ has a quadratic in the denominator and a linear term in the numerator. This leads us to try substitution. Let $u = x^2+6x+11$, so $du = (2x+6)\ dx$. The numerator is $2x-1$, not $2x+6$, but we can get a $2x+6$ term in the numerator by adding 0 in the form of "$7-7$."

$$\begin{aligned}\frac{2x-1}{x^2+6x+11} &= \frac{2x-1+7-7}{x^2+6x+11} \\ &= \frac{2x+6}{x^2+6x+11} - \frac{7}{x^2+6x+11}.\end{aligned}$$

We can now integrate the first term with substitution, leading to a $\ln|x^2+6x+11|$ term. The final term can be integrated using arctangent. First, complete the square in the denominator:

$$\frac{7}{x^2+6x+11} = \frac{7}{(x+3)^2+2}.$$

An antiderivative of the latter term can be found using Theorem 2.1.3 and substitution:

$$\int \frac{7}{x^2+6x+11}\ dx = \frac{7}{\sqrt{2}}\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right) + C.$$

Let's start at the beginning and put all of the steps together.

$$\begin{aligned}\int \frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\ dx &= \int \left(\frac{5}{x+1} + \frac{2x-1}{x^2+6x+11}\right)\ dx \\ &= \int \frac{5}{x+1}\ dx + \int \frac{2x+6}{x^2+6x+11}\ dx - \int \frac{7}{x^2+6x+11}\ dx \\ &= 5\ln|x+1| + \ln|x^2+6x+11| - \frac{7}{\sqrt{2}}\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right) + C.\end{aligned}$$

As with many other problems in calculus, it is important to remember that one is not expected to "see" the final answer immediately after seeing the problem. Rather, given the initial problem, we break it down into smaller problems that are easier to solve. The final answer is a combination of the answers of the smaller problems.

Partial Fraction Decomposition is an important tool when dealing with rational functions. Note that at its heart, it is a technique of algebra, not calculus, as we are rewriting a fraction in a new form. Regardless, it is very useful in the realm of calculus as it lets us evaluate a certain set of "complicated" integrals.

Notes:

The next section introduces new functions, called the Hyperbolic Functions. They will allow us to make substitutions similar to those found when studying Trigonometric Substitution, allowing us to approach even more integration problems.

Notes:

Exercises 2.5

Terms and Concepts

1. Fill in the blank: Partial Fraction Decomposition is a method of rewriting ________ functions.

2. T/F: It is sometimes necessary to use polynomial division before using Partial Fraction Decomposition.

3. Decompose $\dfrac{1}{x^2-3x}$ without solving for the coefficients, as done in Example 2.5.1.

4. Decompose $\dfrac{7-x}{x^2-9}$ without solving for the coefficients, as done in Example 2.5.1.

5. Decompose $\dfrac{x-3}{x^2-7}$ without solving for the coefficients, as done in Example 2.5.1.

6. Decompose $\dfrac{2x+5}{x^3+7x}$ without solving for the coefficients, as done in Example 2.5.1.

Problems

In Exercises 7 – 25, evaluate the indefinite integral.

7. $\displaystyle\int \frac{7x+7}{x^2+3x-10}\,dx$

8. $\displaystyle\int \frac{7x-2}{x^2+x}\,dx$

9. $\displaystyle\int \frac{-4}{3x^2-12}\,dx$

10. $\displaystyle\int \frac{x+7}{(x+5)^2}\,dx$

11. $\displaystyle\int \frac{-3x-20}{(x+8)^2}\,dx$

12. $\displaystyle\int \frac{9x^2+11x+7}{x(x+1)^2}\,dx$

13. $\displaystyle\int \frac{-12x^2-x+33}{(x-1)(x+3)(3-2x)}\,dx$

14. $\displaystyle\int \frac{94x^2-10x}{(7x+3)(5x-1)(3x-1)}\,dx$

15. $\displaystyle\int \frac{x^2+x+1}{x^2+x-2}\,dx$

16. $\displaystyle\int \frac{x^3}{x^2-x-20}\,dx$

17. $\displaystyle\int \frac{2x^2-4x+6}{x^2-2x+3}\,dx$

18. $\displaystyle\int \frac{1}{x^3+2x^2+3x}\,dx$

19. $\displaystyle\int \frac{x^2+x+5}{x^2+4x+10}\,dx$

20. $\displaystyle\int \frac{12x^2+21x+3}{(x+1)(3x^2+5x-1)}\,dx$

21. $\displaystyle\int \frac{6x^2+8x-4}{(x-3)(x^2+6x+10)}\,dx$

22. $\displaystyle\int \frac{2x^2+x+1}{(x+1)(x^2+9)}\,dx$

23. $\displaystyle\int \frac{x^2-20x-69}{(x-7)(x^2+2x+17)}\,dx$

24. $\displaystyle\int \frac{9x^2-60x+33}{(x-9)(x^2-2x+11)}\,dx$

25. $\displaystyle\int \frac{6x^2+45x+121}{(x+2)(x^2+10x+27)}\,dx$

In Exercises 26 – 29, evaluate the definite integral.

26. $\displaystyle\int_1^2 \frac{8x+21}{(x+2)(x+3)}\,dx$

27. $\displaystyle\int_0^5 \frac{14x+6}{(3x+2)(x+4)}\,dx$

28. $\displaystyle\int_{-1}^1 \frac{x^2+5x-5}{(x-10)(x^2+4x+5)}\,dx$

29. $\displaystyle\int_0^1 \frac{x}{(x+1)(x^2+2x+1)}\,dx$

2.6 Improper Integration

We begin this section by considering the following definite integrals:

- $\displaystyle\int_0^{100} \frac{1}{1+x^2}\, dx \approx 1.5608,$
- $\displaystyle\int_0^{1000} \frac{1}{1+x^2}\, dx \approx 1.5698,$
- $\displaystyle\int_0^{10,000} \frac{1}{1+x^2}\, dx \approx 1.5707.$

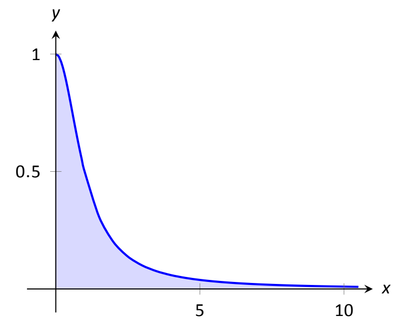


Figure 2.13: Graphing $f(x) = \dfrac{1}{1+x^2}$.

Notice how the integrand is $1/(1+x^2)$ in each integral (which is sketched in Figure 2.13). As the upper bound gets larger, one would expect the "area under the curve" would also grow. While the definite integrals do increase in value as the upper bound grows, they are not increasing by much. In fact, consider:

$$\int_0^b \frac{1}{1+x^2}\, dx = \tan^{-1} x\Big|_0^b = \tan^{-1} b - \tan^{-1} 0 = \tan^{-1} b.$$

As $b \to \infty$, $\tan^{-1} b \to \pi/2$. Therefore it seems that as the upper bound b grows, the value of the definite integral $\displaystyle\int_0^b \frac{1}{1+x^2}\, dx$ approaches $\pi/2 \approx 1.5708$. This should strike the reader as being a bit amazing: even though the curve extends "to infinity," it has a finite amount of area underneath it.

When we defined the definite integral $\displaystyle\int_a^b f(x)\, dx$, we made two stipulations:

1. The interval over which we integrated, $[a, b]$, was a finite interval, and
2. The function $f(x)$ was continuous on $[a, b]$ (ensuring that the range of f was finite).

In this section we consider integrals where one or both of the above conditions do not hold. Such integrals are called **improper integrals.**

Notes:

Improper Integrals with Infinite Bounds

Definition 2.6.1 Improper Integrals with Infinite Bounds; Converge, Diverge

1. Let f be a continuous function on $[a,\infty)$. Define

$$\int_a^\infty f(x)\,dx \quad \text{to be} \quad \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

2. Let f be a continuous function on $(-\infty,b]$. Define

$$\int_{-\infty}^b f(x)\,dx \quad \text{to be} \quad \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

3. Let f be a continuous function on $(-\infty,\infty)$. Let c be any real number; define

$$\int_{-\infty}^\infty f(x)\,dx \quad \text{to be} \quad \lim_{a\to-\infty}\int_a^c f(x)\,dx \;+\; \lim_{b\to\infty}\int_c^b f(x)\,dx.$$

An improper integral is said to **converge** if its corresponding limit exists; otherwise, it **diverges**. The improper integral in part 3 converges if and only if both of its limits exist.

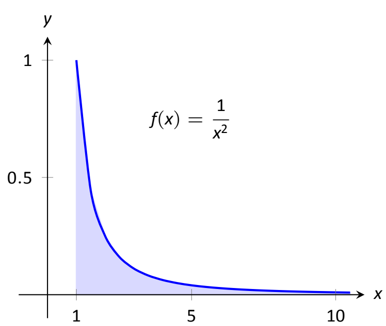


Figure 2.14: A graph of $f(x) = \frac{1}{x^2}$ in Example 2.6.1.

Example 2.6.1 Evaluating improper integrals

Evaluate the following improper integrals.

1. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$
2. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$
3. $\displaystyle\int_{-\infty}^0 e^x\,dx$
4. $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$

Solution

1.
$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \frac{-1}{x}\bigg|_1^b$$
$$= \lim_{b\to\infty}\frac{-1}{b} + 1$$
$$= 1.$$

A graph of the area defined by this integral is given in Figure 2.14.

Notes:

2.
$$\begin{aligned}\int_1^\infty \frac{1}{x}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx\\ &= \lim_{b\to\infty} \ln|x|\Big|_1^b\\ &= \lim_{b\to\infty} \ln(b)\\ &= \infty.\end{aligned}$$

The limit does not exist, hence the improper integral $\int_1^\infty \frac{1}{x}\,dx$ diverges. Compare the graphs in Figures 2.14 and 2.15; notice how the graph of $f(x) = 1/x$ is noticeably larger. This difference is enough to cause the improper integral to diverge.

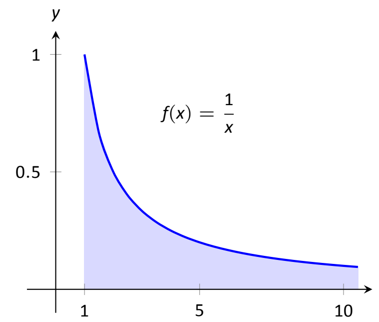


Figure 2.15: A graph of $f(x) = \frac{1}{x}$ in Example 2.6.1.

3.
$$\begin{aligned}\int_{-\infty}^0 e^x\,dx &= \lim_{a\to-\infty}\int_a^0 e^x\,dx\\ &= \lim_{a\to-\infty} e^x\Big|_a^0\\ &= \lim_{a\to-\infty} e^0 - e^a\\ &= 1.\end{aligned}$$

A graph of the area defined by this integral is given in Figure 2.16.

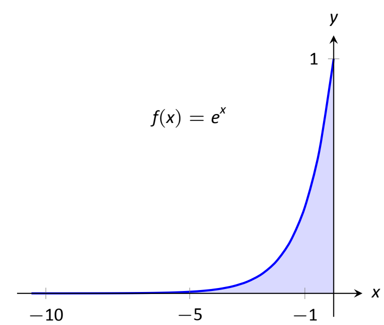


Figure 2.16: A graph of $f(x) = e^x$ in Example 2.6.1.

4. We will need to break this into two improper integrals and choose a value of c as in part 3 of Definition 2.6.1. Any value of c is fine; we choose $c = 0$.

$$\begin{aligned}\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx &= \lim_{a\to-\infty}\int_a^0 \frac{1}{1+x^2}\,dx + \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx\\ &= \lim_{a\to-\infty} \tan^{-1}x\Big|_a^0 + \lim_{b\to\infty} \tan^{-1}x\Big|_0^b\\ &= \lim_{a\to-\infty}\left(\tan^{-1}0 - \tan^{-1}a\right) + \lim_{b\to\infty}\left(\tan^{-1}b - \tan^{-1}0\right)\\ &= \left(0 - \frac{-\pi}{2}\right) + \left(\frac{\pi}{2} - 0\right).\end{aligned}$$

Each limit exists, hence the original integral converges and has value:

$$= \pi.$$

A graph of the area defined by this integral is given in Figure 2.17.

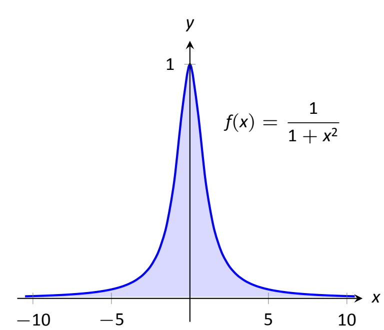


Figure 2.17: A graph of $f(x) = \frac{1}{1+x^2}$ in Example 2.6.1.

Notes:

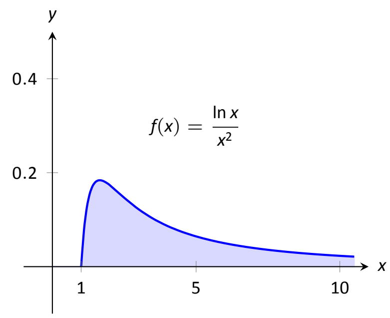


Figure 2.18: A graph of $f(x) = \frac{\ln x}{x^2}$ in Example 2.6.2.

The previous section introduced l'Hôpital's Rule, a method of evaluating limits that return indeterminate forms. It is not uncommon for the limits resulting from improper integrals to need this rule as demonstrated next.

Example 2.6.2 Improper integration and l'Hôpital's Rule

Evaluate the improper integral $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$.

Solution This integral will require the use of Integration by Parts. Let $u = \ln x$ and $dv = 1/x^2\,dx$. Then

$$\begin{aligned}\int_1^\infty \frac{\ln x}{x^2}\,dx &= \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx\\ &= \lim_{b\to\infty}\left(-\frac{\ln x}{x}\Big|_1^b + \int_1^b \frac{1}{x^2}\,dx\right)\\ &= \lim_{b\to\infty}\left(-\frac{\ln x}{x} - \frac{1}{x}\right)\Big|_1^b\\ &= \lim_{b\to\infty}\left(-\frac{\ln b}{b} - \frac{1}{b} - (-\ln 1 - 1)\right).\end{aligned}$$

The $1/b$ and $\ln 1$ terms go to 0, leaving $\displaystyle\lim_{b\to\infty} -\frac{\ln b}{b} + 1$. We need to evaluate $\displaystyle\lim_{b\to\infty}\frac{\ln b}{b}$ with l'Hôpital's Rule. We have:

$$\begin{aligned}\lim_{b\to\infty}\frac{\ln b}{b} &\overset{\text{by LHR}}{=} \lim_{b\to\infty}\frac{1/b}{1}\\ &= 0.\end{aligned}$$

Thus the improper integral evaluates as:

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = 1.$$

Improper Integrals with Infinite Range

We have just considered definite integrals where the interval of integration was infinite. We now consider another type of improper integration, where the range of the integrand is infinite.

Notes:

Definition 2.6.2 Improper Integration with Infinite Range

Let $f(x)$ be a continuous function on $[a,b]$ except at c, $a \le c \le b$, where $x=c$ is a vertical asymptote of f. Define

$$\int_a^b f(x)\ dx = \lim_{t\to c^-}\int_a^t f(x)\ dx + \lim_{t\to c^+}\int_t^b f(x)\ dx.$$

Note: In Definition 2.6.2, c can be one of the endpoints (a or b). In that case, there is only one limit to consider as part of the definition.

Example 2.6.3 Improper integration of functions with infinite range

Evaluate the following improper integrals:

1. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\ dx$
2. $\displaystyle\int_{-1}^1 \frac{1}{x^2}\ dx.$

Solution

1. A graph of $f(x) = 1/\sqrt{x}$ is given in Figure 2.19. Notice that f has a vertical asymptote at $x=0$; in some sense, we are trying to compute the area of a region that has no "top." Could this have a finite value?

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}}\ dx &= \lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\ dx\\ &= \lim_{a\to 0^+} 2\sqrt{x}\Big|_a^1\\ &= \lim_{a\to 0^+} 2\left(\sqrt{1}-\sqrt{a}\right)\\ &= 2.\end{aligned}$$

It turns out that the region does have a finite area even though it has no upper bound (strange things can occur in mathematics when considering the infinite).

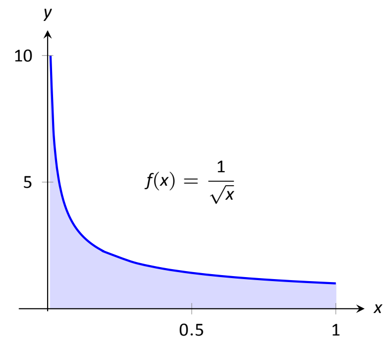


Figure 2.19: A graph of $f(x) = \frac{1}{\sqrt{x}}$ in Example 2.6.3.

2. The function $f(x) = 1/x^2$ has a vertical asymptote at $x=0$, as shown in Figure 2.20, so this integral is an improper integral. Let's eschew using limits for a moment and proceed without recognizing

Notes:

the improper nature of the integral. This leads to:

$$\begin{aligned}\int_{-1}^{1} \frac{1}{x^2}\ dx &= -\frac{1}{x}\Big|_{-1}^{1} \\ &= -1-(1) \\ &= -2!\end{aligned}$$

Clearly the area in question is above the x-axis, yet the area is supposedly negative! Why does our answer not match our intuition? To answer this, evaluate the integral using Definition 2.6.2.

$$\begin{aligned}\int_{-1}^{1} \frac{1}{x^2}\ dx &= \lim_{t\to 0^-}\int_{-1}^{t} \frac{1}{x^2}\ dx + \lim_{t\to 0^+}\int_{t}^{1} \frac{1}{x^2}\ dx \\ &= \lim_{t\to 0^-} -\frac{1}{x}\Big|_{-1}^{t} + \lim_{t\to 0^+} -\frac{1}{x}\Big|_{t}^{1} \\ &= \lim_{t\to 0^-} -\frac{1}{t} - 1 + \lim_{t\to 0^+} -1 + \frac{1}{t} \\ &\Rightarrow \Big(\infty - 1\Big) \ + \ \Big(-1+\infty\Big).\end{aligned}$$

Neither limit converges hence the original improper integral diverges. The nonsensical answer we obtained by ignoring the improper nature of the integral is just that: nonsensical.

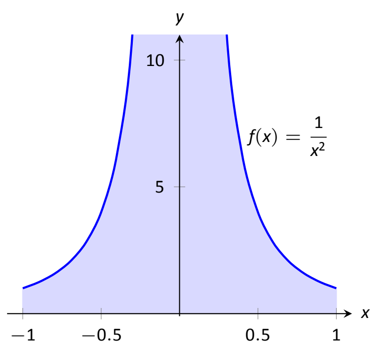


Figure 2.20: A graph of $f(x) = \frac{1}{x^2}$ in Example 2.6.3.

Understanding Convergence and Divergence

Oftentimes we are interested in knowing simply whether or not an improper integral converges, and not necessarily the value of a convergent integral. We provide here several tools that help determine the convergence or divergence of improper integrals without integrating.

Our first tool is to understand the behavior of functions of the form $\dfrac{1}{x^p}$.

Example 2.6.4 Improper integration of $1/x^p$

Determine the values of p for which $\displaystyle\int_1^\infty \frac{1}{x^p}\ dx$ converges.

Notes:

Solution We begin by integrating and then evaluating the limit.

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx\\ &= \lim_{b\to\infty}\int_1^b x^{-p}\,dx \qquad \text{(assume } p\neq 1\text{)}\\ &= \lim_{b\to\infty}\frac{1}{-p+1}x^{-p+1}\Big|_1^b\\ &= \lim_{b\to\infty}\frac{1}{1-p}\big(b^{1-p}-1^{1-p}\big).\end{aligned}$$

When does this limit converge – i.e., when is this limit *not* ∞? This limit converges precisely when the power of b is less than 0: when $1-p<0 \Rightarrow 1<p$.

Our analysis shows that if $p>1$, then $\int_1^\infty \frac{1}{x^p}\,dx$ converges. When $p<1$ the improper integral diverges; we showed in Example 2.6.1 that when $p=1$ the integral also diverges.

Figure 2.21 graphs $y=1/x$ with a dashed line, along with graphs of $y=1/x^p$, $p<1$, and $y=1/x^q$, $q>1$. Somehow the dashed line forms a dividing line between convergence and divergence.

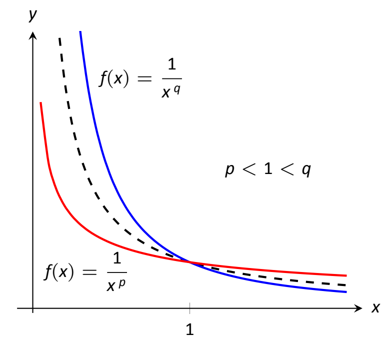


Figure 2.21: Plotting functions of the form $1/x^p$ in Example 2.6.4.

The result of Example 2.6.4 provides an important tool in determining the convergence of other integrals. A similar result is proved in the exercises about improper integrals of the form $\int_0^1 \frac{1}{x^p}\,dx$. These results are summarized in the following Key Idea.

Key Idea 2.6.1 Convergence of Improper Integrals $\int_1^\infty \frac{1}{x^p}\,dx$ and $\int_0^1 \frac{1}{x^p}\,dx$.

1. The improper integral $\int_1^\infty \frac{1}{x^p}\,dx$ converges when $p>1$ and diverges when $p\leq 1$.
2. The improper integral $\int_0^1 \frac{1}{x^p}\,dx$ converges when $p<1$ and diverges when $p\geq 1$.

Notes:

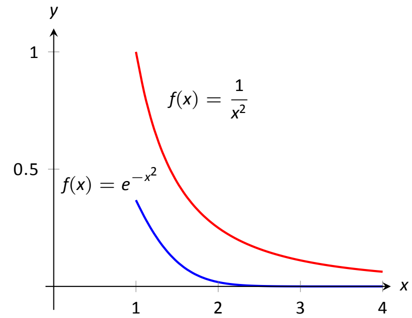


Figure 2.22: Graphs of $f(x) = e^{-x^2}$ and $f(x) = 1/x^2$ in Example 2.6.5.

A basic technique in determining convergence of improper integrals is to compare an integrand whose convergence is unknown to an integrand whose convergence is known. We often use integrands of the form $1/x^p$ to compare to as their convergence on certain intervals is known. This is described in the following theorem.

Theorem 2.6.1 Direct Comparison Test for Improper Integrals

Let f and g be continuous on $[a, \infty)$ where $0 \le f(x) \le g(x)$ for all x in $[a, \infty)$.

1. If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then $\displaystyle\int_a^\infty f(x)\,dx$ converges.
2. If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then $\displaystyle\int_a^\infty g(x)\,dx$ diverges.

Example 2.6.5 Determining convergence of improper integrals

Determine the convergence of the following improper integrals.

1. $\displaystyle\int_1^\infty e^{-x^2}\,dx$ 2. $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^2-x}}\,dx$

Solution

1. The function $f(x) = e^{-x^2}$ does not have an antiderivative expressible in terms of elementary functions, so we cannot integrate directly. It is comparable to $g(x) = 1/x^2$, and as demonstrated in Figure 2.22, $e^{-x^2} < 1/x^2$ on $[1, \infty)$. We know from Key Idea 2.6.1 that $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges, hence $\displaystyle\int_1^\infty e^{-x^2}\,dx$ also converges.

2. Note that for large values of x, $\displaystyle\frac{1}{\sqrt{x^2-x}} \approx \frac{1}{\sqrt{x^2}} = \frac{1}{x}$. We know from Key Idea 2.6.1 and the subsequent note that $\displaystyle\int_3^\infty \frac{1}{x}\,dx$ diverges, so we seek to compare the original integrand to $1/x$.

   It is easy to see that when $x > 0$, we have $x = \sqrt{x^2} > \sqrt{x^2-x}$.

Note: We used the upper and lower bound of "1" in Key Idea 2.6.1 for convenience. It can be replaced by any a where $a > 0$.

Notes:

Taking reciprocals reverses the inequality, giving

$$\frac{1}{x} < \frac{1}{\sqrt{x^2-x}}.$$

Using Theorem 2.6.1, we conclude that since $\int_3^\infty \frac{1}{x}\,dx$ diverges, $\int_3^\infty \frac{1}{\sqrt{x^2-x}}\,dx$ diverges as well. Figure 2.23 illustrates this.

Being able to compare "unknown" integrals to "known" integrals is very useful in determining convergence. However, some of our examples were a little "too nice." For instance, it was convenient that $\frac{1}{x} < \frac{1}{\sqrt{x^2-x}}$, but what if the "$-x$" were replaced with a "$+2x+5$"? That is, what can we say about the convergence of $\int_3^\infty \frac{1}{\sqrt{x^2+2x+5}}\,dx$? We have $\frac{1}{x} > \frac{1}{\sqrt{x^2+2x+5}}$, so we cannot use Theorem 2.6.1.

In cases like this (and many more) it is useful to employ the following theorem.

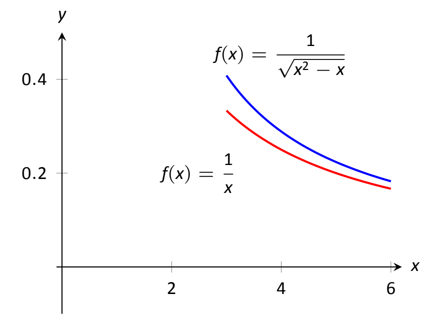


Figure 2.23: Graphs of $f(x) = 1/\sqrt{x^2-x}$ and $f(x) = 1/x$ in Example 2.6.5.

Theorem 2.6.2 Limit Comparison Test for Improper Integrals

Let f and g be continuous functions on $[a,\infty)$ where $f(x) > 0$ and $g(x) > 0$ for all x. If

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

either both converge or both diverge.

Example 2.6.6 Determining convergence of improper integrals

Determine the convergence of $\int_3^\infty \frac{1}{\sqrt{x^2+2x+5}}\,dx$.

Solution As x gets large, the quadratic inside the square root function will begin to behave much like $y = x$. So we compare $\frac{1}{\sqrt{x^2+2x+5}}$

Notes:

to $\dfrac{1}{x}$ with the Limit Comparison Test:

$$\lim_{x\to\infty} \frac{1/\sqrt{x^2+2x+5}}{1/x} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+2x+5}}.$$

The immediate evaluation of this limit returns ∞/∞, an indeterminate form. Using l'Hôpital's Rule seems appropriate, but in this situation, it does not lead to useful results. (We encourage the reader to employ l'Hôpital's Rule at least once to verify this.)

The trouble is the square root function. To get rid of it, we employ the following fact: If $\lim\limits_{x\to c} f(x) = L$, then $\lim\limits_{x\to c} f(x)^2 = L^2$. (This is true when either c or L is ∞.) So we consider now the limit

$$\lim_{x\to\infty} \frac{x^2}{x^2+2x+5}.$$

This converges to 1, meaning the original limit also converged to 1. As x gets very large, the function $\dfrac{1}{\sqrt{x^2+2x+5}}$ looks very much like $\dfrac{1}{x}$. Since we know that $\displaystyle\int_3^\infty \frac{1}{x}\,dx$ diverges, by the Limit Comparison Test we know that $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^2+2x+5}}\,dx$ also diverges. Figure 2.24 graphs $f(x) = 1/\sqrt{x^2+2x+5}$ and $f(x) = 1/x$, illustrating that as x gets large, the functions become indistinguishable.

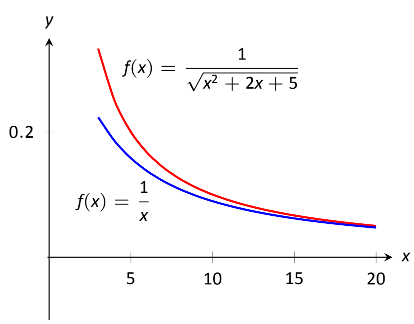


Figure 2.24: Graphing $f(x) = \frac{1}{\sqrt{x^2+2x+5}}$ and $f(x) = \frac{1}{x}$ in Example 2.6.6.

Both the Direct and Limit Comparison Tests were given in terms of integrals over an infinite interval. There are versions that apply to improper integrals with an infinite range, but as they are a bit wordy and a little more difficult to employ, they are omitted from this text.

This chapter has explored many integration techniques. We learned Substitution, which "undoes" the Chain Rule of differentiation, as well as Integration by Parts, which "undoes" the Product Rule. We learned specialized techniques for handling trigonometric functions and introduced the hyperbolic functions, which are closely related to the trigonometric functions. All techniques effectively have this goal in common: rewrite the integrand in a new way so that the integration step is easier to see and implement.

As stated before, integration is, in general, hard. It is easy to write a function whose antiderivative is impossible to write in terms of elementary functions, and even when a function does have an antiderivative expressible by elementary functions, it may be really hard to discover what it is.

Notes:

The powerful computer algebra system *Mathematica*® has approximately 1,000 pages of code dedicated to integration.

Do not let this difficulty discourage you. There is great value in learning integration techniques, as they allow one to manipulate an integral in ways that can illuminate a concept for greater understanding. There is also great value in understanding the need for good numerical techniques: the Trapezoidal and Simpson's Rules are just the beginning of powerful techniques for approximating the value of integration.

The next chapter stresses the uses of integration. We generally do not find antiderivatives for antiderivative's sake, but rather because they provide the solution to some type of problem. The following chapter introduces us to a number of different problems whose solution is provided by integration.

Notes:

Exercises 2.6

Terms and Concepts

1. The definite integral was defined with what two stipulations?

2. If $\lim\limits_{b\to\infty}\int_0^b f(x)\,dx$ exists, then the integral $\int_0^\infty f(x)\,dx$ is said to ______________.

3. If $\int_1^\infty f(x)\,dx = 10$, and $0 \le g(x) \le f(x)$ for all x, then we know that $\int_1^\infty g(x)\,dx$ ______________.

4. For what values of p will $\int_1^\infty \frac{1}{x^p}\,dx$ converge?

5. For what values of p will $\int_{10}^\infty \frac{1}{x^p}\,dx$ converge?

6. For what values of p will $\int_0^1 \frac{1}{x^p}\,dx$ converge?

Problems

In Exercises 7 – 33, evaluate the given improper integral.

7. $\int_0^\infty e^{5-2x}\,dx$

8. $\int_1^\infty \frac{1}{x^3}\,dx$

9. $\int_1^\infty x^{-4}\,dx$

10. $\int_{-\infty}^\infty \frac{1}{x^2+9}\,dx$

11. $\int_{-\infty}^0 2^x\,dx$

12. $\int_{-\infty}^0 \left(\frac{1}{2}\right)^x\,dx$

13. $\int_{-\infty}^\infty \frac{x}{x^2+1}\,dx$

14. $\int_{-\infty}^\infty \frac{x}{x^2+4}\,dx$

15. $\int_2^\infty \frac{1}{(x-1)^2}\,dx$

16. $\int_1^2 \frac{1}{(x-1)^2}\,dx$

17. $\int_2^\infty \frac{1}{x-1}\,dx$

18. $\int_1^2 \frac{1}{x-1}\,dx$

19. $\int_{-1}^1 \frac{1}{x}\,dx$

20. $\int_1^3 \frac{1}{x-2}\,dx$

21. $\int_0^\pi \sec^2 x\,dx$

22. $\int_{-2}^1 \frac{1}{\sqrt{|x|}}\,dx$

23. $\int_0^\infty xe^{-x}\,dx$

24. $\int_0^\infty xe^{-x^2}\,dx$

25. $\int_{-\infty}^\infty xe^{-x^2}\,dx$

26. $\int_{-\infty}^\infty \frac{1}{e^x+e^{-x}}\,dx$

27. $\int_0^1 x\ln x\,dx$

28. $\int_1^\infty \frac{\ln x}{x}\,dx$

29. $\int_0^1 \ln x\,dx$

30. $\int_1^\infty \frac{\ln x}{x^2}\,dx$

31. $\int_1^\infty \frac{\ln x}{\sqrt{x}}\,dx$

32. $\int_0^\infty e^{-x}\sin x\,dx$

33. $\int_0^\infty e^{-x}\cos x\,dx$

In Exercises 34 – 43, use the Direct Comparison Test or the Limit Comparison Test to determine whether the given definite integral converges or diverges. Clearly state what test is being used and what function the integrand is being compared to.

34. $\displaystyle\int_{10}^{\infty} \frac{3}{\sqrt{3x^2+2x-5}}\ dx$

35. $\displaystyle\int_{2}^{\infty} \frac{4}{\sqrt{7x^3-x}}\ dx$

36. $\displaystyle\int_{0}^{\infty} \frac{\sqrt{x+3}}{\sqrt{x^3-x^2+x+1}}\ dx$

37. $\displaystyle\int_{1}^{\infty} e^{-x}\ln x\ dx$

38. $\displaystyle\int_{5}^{\infty} e^{-x^2+3x+1}\ dx$

39. $\displaystyle\int_{0}^{\infty} \frac{\sqrt{x}}{e^x}\ dx$

40. $\displaystyle\int_{2}^{\infty} \frac{1}{x^2+\sin x}\ dx$

41. $\displaystyle\int_{0}^{\infty} \frac{x}{x^2+\cos x}\ dx$

42. $\displaystyle\int_{0}^{\infty} \frac{1}{x+e^x}\ dx$

43. $\displaystyle\int_{0}^{\infty} \frac{1}{e^x-x}\ dx$

2.7 Numerical Integration

The Fundamental Theorem of Calculus gives a concrete technique for finding the exact value of a definite integral. That technique is based on computing antiderivatives. Despite the power of this theorem, there are still situations where we must *approximate* the value of the definite integral instead of finding its exact value. The first situation we explore is where we *cannot* compute the antiderivative of the integrand. The second case is when we actually do not know the integrand, but only its value when evaluated at certain points.

An **elementary function** is any function that is a combination of polynomials, n^{th} roots, rational, exponential, logarithmic and trigonometric functions. We can compute the derivative of any elementary function, but there are many elementary functions of which we cannot compute an antiderivative. For example, the following functions do not have antiderivatives that we can express with elementary functions:

$$e^{-x^2}, \quad \sin(x^3) \quad \text{and} \quad \frac{\sin x}{x}.$$

The simplest way to refer to the antiderivatives of e^{-x^2} is to simply write $\int e^{-x^2}\,dx$.

This section outlines three common methods of approximating the value of definite integrals. We describe each as a systematic method of approximating area under a curve. By approximating this area accurately, we find an accurate approximation of the corresponding definite integral.

We will apply the methods we learn in this section to the following definite integrals:

$$\int_0^1 e^{-x^2}\,dx, \quad \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx, \quad \text{and} \quad \int_{0.5}^{4\pi} \frac{\sin(x)}{x}\,dx,$$

as pictured in Figure 2.25.

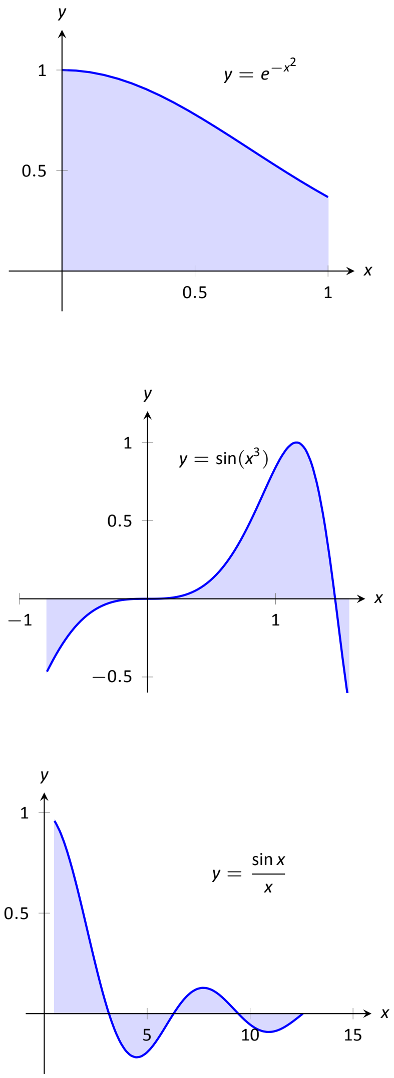


Figure 2.25: Graphically representing three definite integrals that cannot be evaluated using antiderivatives.

The Left and Right Hand Rule Methods

In Section 1.3 we addressed the problem of evaluating definite integrals by approximating the area under the curve using rectangles. We revisit those ideas here before introducing other methods of approximating definite integrals.

We start with a review of notation. Let f be a continuous function on the interval $[a,b]$. We wish to approximate $\displaystyle\int_a^b f(x)\,dx$. We partition

Notes:

$[a, b]$ into n equally spaced subintervals, each of length $\Delta x = \dfrac{b-a}{n}$. The endpoints of these subintervals are labeled as

$$x_1 = a,\ x_2 = a+\Delta x,\ x_3 = a+2\Delta x,\ \ldots,\ x_i = a+(i-1)\Delta x,\ \ldots,\ x_{n+1} = b.$$

Key Idea 1.3.1 states that to use the Left Hand Rule we use the summation $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x$ and to use the Right Hand Rule we use $\displaystyle\sum_{i=1}^{n} f(x_{i+1})\Delta x$. We review the use of these rules in the context of examples.

Example 2.7.1 Approximating definite integrals with rectangles

Approximate $\displaystyle\int_0^1 e^{-x^2}\,dx$ using the Left and Right Hand Rules with 5 equally spaced subintervals.

Solution We begin by partitioning the interval $[0, 1]$ into 5 equally spaced intervals. We have $\Delta x = \frac{1-0}{5} = 1/5 = 0.2$, so

$$x_1 = 0,\ x_2 = 0.2,\ x_3 = 0.4,\ x_4 = 0.6,\ x_5 = 0.8,\ \text{and}\ x_6 = 1.$$

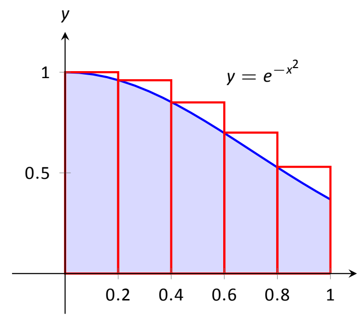
Using the Left Hand Rule, we have:

$$\begin{aligned}\sum_{i=1}^{n} f(x_i)\Delta x &= \big(f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)\big)\Delta x \\ &= \big(f(0) + f(0.2) + f(0.4) + f(0.6) + f(0.8)\big)\Delta x \\ &\approx \big(1 + 0.961 + 0.852 + 0.698 + 0.527)(0.2) \\ &\approx 0.808.\end{aligned}$$

Using the Right Hand Rule, we have:

$$\begin{aligned}\sum_{i=1}^{n} f(x_{i+1})\Delta x &= \big(f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6)\big)\Delta x \\ &= \big(f(0.2) + f(0.4) + f(0.6) + f(0.8) + f(1)\big)\Delta x \\ &\approx \big(0.961 + 0.852 + 0.698 + 0.527 + 0.368)(0.2) \\ &\approx 0.681.\end{aligned}$$

Figure 2.26 shows the rectangles used in each method to approximate the definite integral. These graphs show that in this particular case, the Left Hand Rule is an over approximation and the Right Hand Rule is an



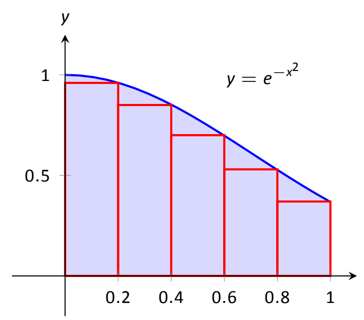


Figure 2.26: Approximating $\int_0^1 e^{-x^2}\,dx$ in Example 2.7.1.

Notes:

x_i	Exact	Approx.	$\sin(x_i^3)$
x_1	$-\pi/4$	-0.785	-0.466
x_2	$-7\pi/40$	-0.550	-0.165
x_3	$-\pi/10$	-0.314	-0.031
x_4	$-\pi/40$	-0.0785	0
x_5	$\pi/20$	0.157	0.004
x_6	$\pi/8$	0.393	0.061
x_7	$\pi/5$	0.628	0.246
x_8	$11\pi/40$	0.864	0.601
x_9	$7\pi/20$	1.10	0.971
x_{10}	$17\pi/40$	1.34	0.690
x_{11}	$\pi/2$	1.57	-0.670

Figure 2.27: Table of values used to approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\, dx$ in Example 2.7.2.

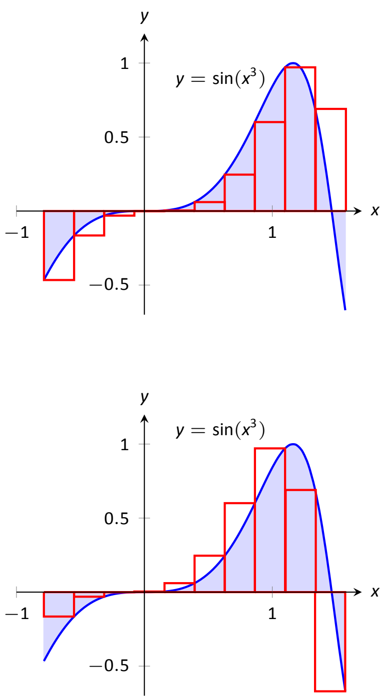


Figure 2.28: Approximating $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\, dx$ in Example 2.7.2.

under approximation. To get a better approximation, we could use more rectangles, as we did in Section 1.3. We could also average the Left and Right Hand Rule results together, giving

$$\frac{0.808 + 0.681}{2} = 0.7445.$$

The actual answer, accurate to 4 places after the decimal, is 0.7468, showing our average is a good approximation.

Example 2.7.2 Approximating definite integrals with rectangles

Approximate $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\, dx$ using the Left and Right Hand Rules with 10 equally spaced subintervals.

Solution We begin by finding Δx:

$$\frac{b-a}{n} = \frac{\pi/2 - (-\pi/4)}{10} = \frac{3\pi}{40} \approx 0.236.$$

It is useful to write out the endpoints of the subintervals in a table; in Figure 2.27, we give the exact values of the endpoints, their decimal approximations, and decimal approximations of $\sin(x^3)$ evaluated at these points.

Once this table is created, it is straightforward to approximate the definite integral using the Left and Right Hand Rules. (Note: the table itself is easy to create, especially with a standard spreadsheet program on a computer. The last two columns are all that are needed.) The Left Hand Rule sums the first 10 values of $\sin(x_i^3)$ and multiplies the sum by Δx; the Right Hand Rule sums the last 10 values of $\sin(x_i^3)$ and multiplies by Δx. Therefore we have:

Left Hand Rule: $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\, dx \approx (1.91)(0.236) = 0.451.$

Right Hand Rule: $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\, dx \approx (1.71)(0.236) = 0.404.$

Average of the Left and Right Hand Rules: 0.4275.

The actual answer, accurate to 3 places after the decimal, is 0.460. Our approximations were once again fairly good. The rectangles used in each approximation are shown in Figure 2.28. It is clear from the graphs that using more rectangles (and hence, narrower rectangles) should result in a more accurate approximation.

The Trapezoidal Rule

Notes:

In Example 2.7.1 we approximated the value of $\int_0^1 e^{-x^2}\,dx$ with 5 rectangles of equal width. Figure 2.26 shows the rectangles used in the Left and Right Hand Rules. These graphs clearly show that rectangles do not match the shape of the graph all that well, and that accurate approximations will only come by using lots of rectangles.

Instead of using rectangles to approximate the area, we can instead use *trapezoids.* In Figure 2.29, we show the region under $f(x) = e^{-x^2}$ on $[0, 1]$ approximated with 5 trapezoids of equal width; the top "corners" of each trapezoid lies on the graph of $f(x)$. It is clear from this figure that these trapezoids more accurately approximate the area under f and hence should give a better approximation of $\int_0^1 e^{-x^2}\,dx$. (In fact, these trapezoids seem to give a *great* approximation of the area!)

The formula for the area of a trapezoid is given in Figure 2.30. We approximate $\int_0^1 e^{-x^2}\,dx$ with these trapezoids in the following example.

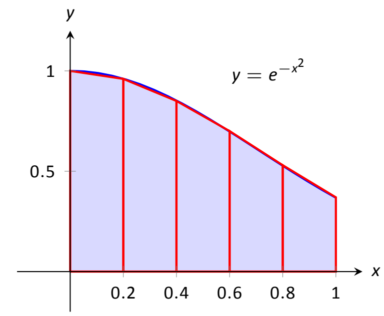


Figure 2.29: Approximating $\int_0^1 e^{-x^2}\,dx$ using 5 trapezoids of equal widths.

Example 2.7.3 Approximating definite integrals using trapezoids

Use 5 trapezoids of equal width to approximate $\displaystyle\int_0^1 e^{-x^2}\,dx$.

Solution To compute the areas of the 5 trapezoids in Figure 2.29, it will again be useful to create a table of values as shown in Figure 2.31.

The leftmost trapezoid has legs of length 1 and 0.961 and a height of 0.2. Thus, by our formula, the area of the leftmost trapezoid is:

$$\frac{1 + 0.961}{2}(0.2) = 0.1961.$$

Moving right, the next trapezoid has legs of length 0.961 and 0.852 and a height of 0.2. Thus its area is:

$$\frac{0.961 + 0.852}{2}(0.2) = 0.1813.$$

The sum of the areas of all 5 trapezoids is:

$$\frac{1 + 0.961}{2}(0.2) + \frac{0.961 + 0.852}{2}(0.2) + \frac{0.852 + 0.698}{2}(0.2) + \frac{0.698 + 0.527}{2}(0.2) + \frac{0.527 + 0.368}{2}(0.2) = 0.7445.$$

We approximate $\displaystyle\int_0^1 e^{-x^2}\,dx \approx 0.7445.$

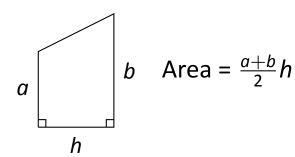


Figure 2.30: The area of a trapezoid.

x_i	$e^{-x_i^2}$
0	1
0.2	0.961
0.4	0.852
0.6	0.698
0.8	0.527
1	0.368

Figure 2.31: A table of values of e^{-x^2}.

Notes:

There are many things to observe in this example. Note how each term in the final summation was multiplied by both 1/2 and by $\Delta x = 0.2$. We can factor these coefficients out, leaving a more concise summation as:

$$\frac{1}{2}(0.2)\Big[(1+0.961)+(0.961+0.852)+(0.852+0.698)+(0.698+0.527)+(0.527+0.368)\Big].$$

Now notice that all numbers except for the first and the last are added twice. Therefore we can write the summation even more concisely as

$$\frac{0.2}{2}\Big[1+2(0.961+0.852+0.698+0.527)+0.368\Big].$$

This is the heart of the **Trapezoidal Rule**, wherein a definite integral $\int_a^b f(x)\ dx$ is approximated by using trapezoids of equal widths to approximate the corresponding area under f. Using n equally spaced subintervals with endpoints $x_1, x_2, \ldots, x_{n+1}$, we again have $\Delta x = \frac{b-a}{n}$. Thus:

$$\begin{aligned}\int_a^b f(x)\ dx &\approx \sum_{i=1}^{n} \frac{f(x_i)+f(x_{i+1})}{2}\Delta x \\ &= \frac{\Delta x}{2}\sum_{i=1}^{n}\big(f(x_i)+f(x_{i+1})\big) \\ &= \frac{\Delta x}{2}\Big[f(x_1)+2\sum_{i=2}^{n} f(x_i)+f(x_{n+1})\Big].\end{aligned}$$

Example 2.7.4 Using the Trapezoidal Rule

Revisit Example 2.7.2 and approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\ dx$ using the Trapezoidal Rule and 10 equally spaced subintervals.

Solution We refer back to Figure 2.27 for the table of values of $\sin(x^3)$. Recall that $\Delta x = 3\pi/40 \approx 0.236$. Thus we have:

$$\begin{aligned}\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\ dx &\approx \frac{0.236}{2}\Big[-0.466+2\Big(-0.165+(-0.031)+\ldots+0.69\Big)+(-0.67)\Big] \\ &= 0.4275.\end{aligned}$$

Notice how "quickly" the Trapezoidal Rule can be implemented once the table of values is created. This is true for all the methods explored

Notes:

in this section; the real work is creating a table of x_i and $f(x_i)$ values. Once this is completed, approximating the definite integral is not difficult. Again, using technology is wise. Spreadsheets can make quick work of these computations and make using lots of subintervals easy.

Also notice the approximations the Trapezoidal Rule gives. It is the average of the approximations given by the Left and Right Hand Rules! This effectively renders the Left and Right Hand Rules obsolete. They are useful when first learning about definite integrals, but if a real approximation is needed, one is generally better off using the Trapezoidal Rule instead of either the Left or Right Hand Rule.

How can we improve on the Trapezoidal Rule, apart from using more and more trapezoids? The answer is clear once we look back and consider what we have *really* done so far. The Left Hand Rule is not *really* about using rectangles to approximate area. Instead, it approximates a function f with constant functions on small subintervals and then computes the definite integral of these constant functions. The Trapezoidal Rule is really approximating a function f with a linear function on a small subinterval, then computes the definite integral of this linear function. In both of these cases the definite integrals are easy to compute in geometric terms.

So we have a progression: we start by approximating f with a constant function and then with a linear function. What is next? A quadratic function. By approximating the curve of a function with lots of parabolas, we generally get an even better approximation of the definite integral. We call this process **Simpson's Rule**, named after Thomas Simpson (1710-1761), even though others had used this rule as much as 100 years prior.

Simpson's Rule

Given one point, we can create a constant function that goes through that point. Given two points, we can create a linear function that goes through those points. Given three points, we can create a quadratic function that goes through those three points (given that no two have the same x–value).

Consider three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) whose x–values are equally spaced and $x_1 < x_2 < x_3$. Let f be the quadratic function that goes through these three points. It is not hard to show that

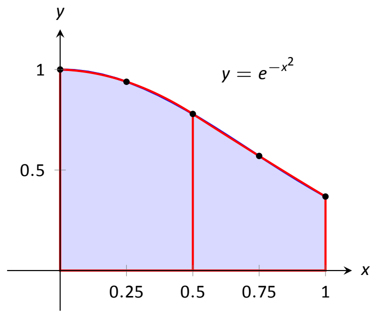
$$\int_{x_1}^{x_3} f(x)\ dx = \frac{x_3 - x_1}{6}\big(y_1 + 4y_2 + y_3\big). \tag{2.3}$$

Consider Figure 2.32. A function f goes through the 3 points shown and the parabola g that also goes through those points is graphed with a

Notes:

x_i	$e^{-x_i^2}$
0	1
0.25	0.939
0.5	0.779
0.75	0.570
1	0.368

(a)



(b)

Figure 2.33: A table of values to approximate $\int_0^1 e^{-x^2}\,dx$, along with a graph of the function.

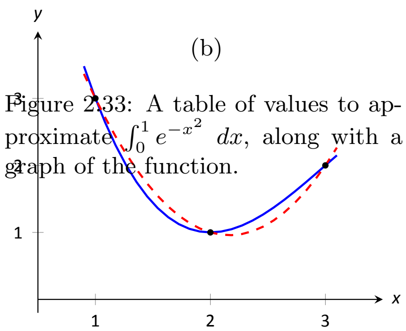


Figure 2.32: A graph of a function f and a parabola that approximates it well on $[1, 3]$.

dashed line. Using our equation from above, we know exactly that

$$\int_1^3 g(x)\,dx = \frac{3-1}{6}\big(3 + 4(1) + 2\big) = 3.$$

Since g is a good approximation for f on $[1, 3]$, we can state that

$$\int_1^3 f(x)\,dx \approx 3.$$

Notice how the interval $[1, 3]$ was split into two subintervals as we needed 3 points. Because of this, whenever we use Simpson's Rule, we need to break the interval into an even number of subintervals.

In general, to approximate $\displaystyle\int_a^b f(x)\,dx$ using Simpson's Rule, subdivide $[a, b]$ into n subintervals, where n is even and each subinterval has width $\Delta x = (b-a)/n$. We approximate f with $n/2$ parabolic curves, using Equation (2.3) to compute the area under these parabolas. Adding up these areas gives the formula:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\Big[f(x_1)+4f(x_2)+2f(x_3)+4f(x_4)+\ldots+2f(x_{n-1})+4f(x_n)+f(x_{n+1})\Big].$$

Note how the coefficients of the terms in the summation have the pattern 1, 4, 2, 4, 2, 4, ..., 2, 4, 1.

Let's demonstrate Simpson's Rule with a concrete example.

Example 2.7.5 Using Simpson's Rule

Approximate $\displaystyle\int_0^1 e^{-x^2}\,dx$ using Simpson's Rule and 4 equally spaced subintervals.

Solution We begin by making a table of values as we have in the past, as shown in Figure 2.33(a). Simpson's Rule states that

$$\int_0^1 e^{-x^2}\,dx \approx \frac{0.25}{3}\Big[1 + 4(0.939) + 2(0.779) + 4(0.570) + 0.368\Big] = 0.7468\overline{3}.$$

Recall in Example 2.7.1 we stated that the correct answer, accurate to 4 places after the decimal, was 0.7468. Our approximation with Simpson's Rule, with 4 subintervals, is better than our approximation with the Trapezoidal Rule using 5!

Figure 2.33(b) shows $f(x) = e^{-x^2}$ along with its approximating parabolas, demonstrating how good our approximation is. The approximating

Notes:

curves are nearly indistinguishable from the actual function.

Example 2.7.6 Using Simpson's Rule

Approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\ dx$ using Simpson's Rule and 10 equally spaced intervals.

Solution Figure 2.34 shows the table of values that we used in the past for this problem, shown here again for convenience. Again, $\Delta x = (\pi/2 + \pi/4)/10 \approx 0.236$.

Simpson's Rule states that

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\ dx \approx \frac{0.236}{3}\Big[(-0.466) + 4(-0.165) + 2(-0.031) + \ldots$$
$$\ldots + 2(0.971) + 4(0.69) + (-0.67)\Big]$$
$$= 0.4701$$

Recall that the actual value, accurate to 3 decimal places, is 0.460. Our approximation is within one $1/100^{\text{th}}$ of the correct value. The graph in Figure 2.35 shows how closely the parabolas match the shape of the graph.

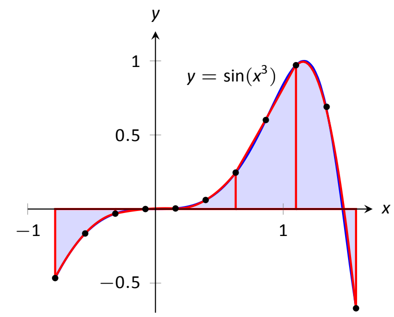


Figure 2.35: Approximating $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\ dx$ in Example 2.7.6 with Simpson's Rule and 10 equally spaced intervals.

x_i	$\sin(x_i^3)$
−0.785	−0.466
−0.550	−0.165
−0.314	−0.031
−0.0785	0
0.157	0.004
0.393	0.061
0.628	0.246
0.864	0.601
1.10	0.971
1.34	0.690
1.57	−0.670

Figure 2.34: Table of values used to approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\ dx$ in Example 2.7.6.

Summary and Error Analysis

We summarize the key concepts of this section thus far in the following Key Idea.

Notes:

Key Idea 2.7.1 Numerical Integration

Let f be a continuous function on $[a, b]$, let n be a positive integer, and let $\Delta x = \dfrac{b-a}{n}$.

Set $x_1 = a$, $x_2 = a + \Delta x$, …, $x_i = a + (i-1)\Delta x$, $x_{n+1} = b$. Consider $\displaystyle\int_a^b f(x)\, dx$.

Left Hand Rule: $\displaystyle\int_a^b f(x)\, dx \approx \Delta x\Big[f(x_1) + f(x_2) + \ldots + f(x_n)\Big]$.

Right Hand Rule: $\displaystyle\int_a^b f(x)\, dx \approx \Delta x\Big[f(x_2) + f(x_3) + \ldots + f(x_{n+1})\Big]$.

Trapezoidal Rule: $\displaystyle\int_a^b f(x)\, dx \approx \frac{\Delta x}{2}\Big[f(x_1)+2f(x_2)+2f(x_3)+\ldots+2f(x_n)+f(x_{n+1})\Big]$.

Simpson's Rule: $\displaystyle\int_a^b f(x)\, dx \approx \frac{\Delta x}{3}\Big[f(x_1)+4f(x_2)+2f(x_3)+\ldots+4f(x_n)+f(x_{n+1})\Big]$ (n even).

In our examples, we approximated the value of a definite integral using a given method then compared it to the "right" answer. This should have raised several questions in the reader's mind, such as:

1. How was the "right" answer computed?
2. If the right answer can be found, what is the point of approximating?
3. If there is value to approximating, how are we supposed to know if the approximation is any good?

These are good questions, and their answers are educational. In the examples, *the* right answer was never computed. Rather, an approximation accurate to a certain number of places after the decimal was given. In Example 2.7.1, we do not know the *exact* answer, but we know it starts with 0.7468. These more accurate approximations were computed using numerical integration but with more precision (i.e., more subintervals and the help of a computer).

Since the exact answer cannot be found, approximation still has its place. How are we to tell if the approximation is any good?

"Trial and error" provides one way. Using technology, make an approximation with, say, 10, 100, and 200 subintervals. This likely will not take much time at all, and a trend should emerge. If a trend does not emerge, try using yet more subintervals. Keep in mind that trial and error is never

Notes:

foolproof; you might stumble upon a problem in which a trend will not emerge.

A second method is to use Error Analysis. While the details are beyond the scope of this text, there are some formulas that give *bounds* for how good your approximation will be. For instance, the formula might state that the approximation is within 0.1 of the correct answer. If the approximation is 1.58, then one knows that the correct answer is between 1.48 and 1.68. By using lots of subintervals, one can get an approximation as accurate as one likes. Theorem 2.7.1 states what these bounds are.

Theorem 2.7.1 Error Bounds in the Trapezoidal and Simpson's Rules

1. Let E_T be the error in approximating $\int_a^b f(x)\,dx$ using the Trapezoidal Rule.

   If f has a continuous 2^{nd} derivative on $[a,b]$ and M is any upper bound of $\left|f''(x)\right|$ on $[a,b]$, then

   $$E_T \leq \frac{(b-a)^3}{12n^2}M.$$

2. Let E_S be the error in approximating $\int_a^b f(x)\,dx$ using Simpson's Rule.

   If f has a continuous 4^{th} derivative on $[a,b]$ and M is any upper bound of $\left|f^{(4)}\right|$ on $[a,b]$, then

   $$E_S \leq \frac{(b-a)^5}{180n^4}M.$$

There are some key things to note about this theorem.

1. The larger the interval, the larger the error. This should make sense intuitively.

2. The error shrinks as more subintervals are used (i.e., as n gets larger).

3. The error in Simpson's Rule has a term relating to the 4^{th} derivative of f. Consider a cubic polynomial: it's 4^{th} derivative is 0. Therefore, the error in approximating the definite integral of a cubic polynomial with Simpson's Rule is 0 – Simpson's Rule computes the exact answer!

Notes:

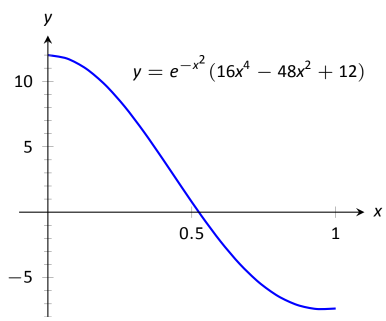


Figure 2.37: Graphing $f^{(4)}(x)$ in Example 2.7.7 to help establish error bounds.

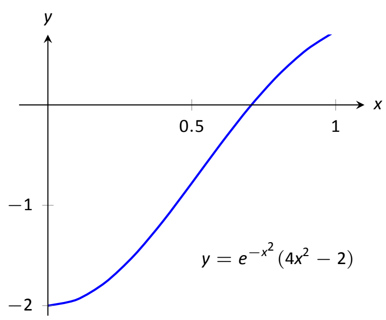


Figure 2.36: Graphing $f''(x)$ in Example 2.7.7 to help establish error bounds.

We revisit Examples 2.7.3 and 2.7.5 and compute the error bounds using Theorem 2.7.1 in the following example.

Example 2.7.7 Computing error bounds

Find the error bounds when approximating $\int_0^1 e^{-x^2}\,dx$ using the Trapezoidal Rule and 5 subintervals, and using Simpson's Rule with 4 subintervals.

Solution

Trapezoidal Rule with $n = 5$:

We start by computing the 2nd derivative of $f(x) = e^{-x^2}$:

$$f''(x) = e^{-x^2}(4x^2 - 2).$$

Figure 2.36 shows a graph of $f''(x)$ on $[0,1]$. It is clear that the largest value of f'', in absolute value, is 2. Thus we let $M = 2$ and apply the error formula from Theorem 2.7.1.

$$E_T = \frac{(1-0)^3}{12 \cdot 5^2} \cdot 2 = 0.00\overline{6}.$$

Our error estimation formula states that our approximation of 0.7445 found in Example 2.7.3 is within 0.0067 of the correct answer, hence we know that

$$0.7445 - 0.0067 = .7378 \le \int_0^1 e^{-x^2}\,dx \le 0.7512 = 0.7445 + 0.0067.$$

We had earlier computed the exact answer, correct to 4 decimal places, to be 0.7468, affirming the validity of Theorem 2.7.1.

Simpson's Rule with $n = 4$:

We start by computing the 4th derivative of $f(x) = e^{-x^2}$:

$$f^{(4)}(x) = e^{-x^2}(16x^4 - 48x^2 + 12).$$

Figure 2.37 shows a graph of $f^{(4)}(x)$ on $[0,1]$. It is clear that the largest value of $f^{(4)}$, in absolute value, is 12. Thus we let $M = 12$ and apply the error formula from Theorem 2.7.1.

$$E_s = \frac{(1-0)^5}{180 \cdot 4^4} \cdot 12 = 0.00026.$$

Notes:

Our error estimation formula states that our approximation of $0.7468\overline{3}$ found in Example 2.7.5 is within 0.00026 of the correct answer, hence we know that

$$0.74683 - 0.00026 = .74657 \leq \int_0^1 e^{-x^2}\ dx \leq 0.74709 = 0.74683 + 0.00026.$$

Once again we affirm the validity of Theorem 2.7.1.

At the beginning of this section we mentioned two main situations where numerical integration was desirable. We have considered the case where an antiderivative of the integrand cannot be computed. We now investigate the situation where the integrand is not known. This is, in fact, the most widely used application of Numerical Integration methods. "Most of the time" we observe behavior but do not know "the" function that describes it. We instead collect data about the behavior and make approximations based off of this data. We demonstrate this in an example.

Example 2.7.8 Approximating distance traveled

One of the authors drove his daughter home from school while she recorded their speed every 30 seconds. The data is given in Figure 2.38. Approximate the distance they traveled.

Solution Recall that by integrating a speed function we get distance traveled. We have information about $v(t)$; we will use Simpson's Rule to approximate $\displaystyle\int_a^b v(t)\ dt$.

The most difficult aspect of this problem is converting the given data into the form we need it to be in. The speed is measured in miles per hour, whereas the time is measured in 30 second increments.

We need to compute $\Delta x = (b-a)/n$. Clearly, $n = 24$. What are a and b? Since we start at time $t = 0$, we have that $a = 0$. The final recorded time came after 24 periods of 30 seconds, which is 12 minutes or 1/5 of an hour. Thus we have

$$\Delta x = \frac{b-a}{n} = \frac{1/5 - 0}{24} = \frac{1}{120}; \qquad \frac{\Delta x}{3} = \frac{1}{360}.$$

Thus the distance traveled is approximately:

$$\begin{aligned}\int_0^{0.2} v(t)\ dt &\approx \frac{1}{360}\Big[f(x_1) + 4f(x_2) + 2f(x_3) + \cdots + 4f(x_n) + f(x_{n+1})\Big] \\ &= \frac{1}{360}\Big[0 + 4\cdot 25 + 2\cdot 22 + \cdots + 2\cdot 40 + 4\cdot 23 + 0\Big] \\ &\approx 6.2167 \text{ miles.}\end{aligned}$$

Time	Speed (mph)
0	0
1	25
2	22
3	19
4	39
5	0
6	43
7	59
8	54
9	51
10	43
11	35
12	40
13	43
14	30
15	0
16	0
17	28
18	40
19	42
20	40
21	39
22	40
23	23
24	0

Figure 2.38: Speed data collected at 30 second intervals for Example 2.7.8.

Notes:

We approximate the author drove 6.2 miles. (Because we are sure the reader wants to know, the author's odometer recorded the distance as about 6.05 miles.)

We started this chapter learning about antiderivatives and indefinite integrals. We then seemed to change focus by looking at areas between the graph of a function and the x-axis. We defined these areas as the definite integral of the function, using a notation very similar to the notation of the indefinite integral. The Fundamental Theorem of Calculus tied these two seemingly separate concepts together: we can find areas under a curve, i.e., we can evaluate a definite integral, using antiderivatives.

We ended the chapter by noting that antiderivatives are sometimes more than difficult to find: they are impossible. Therefore we developed numerical techniques that gave us good approximations of definite integrals.

We used the definite integral to compute areas, and also to compute displacements and distances traveled. There is far more we can do than that. In Chapter 3 we'll see more applications of the definite integral. Before that, in Chapter 2 we'll learn advanced techniques of integration, analogous to learning rules like the Product, Quotient and Chain Rules of differentiation.

Notes:

Exercises 2.7

Terms and Concepts

1. T/F: Simpson's Rule is a method of approximating antiderivatives.

2. What are the two basic situations where approximating the value of a definite integral is necessary?

3. Why are the Left and Right Hand Rules rarely used?

Problems

In Exercises 4 – 11, a definite integral is given.

(a) Approximate the definite integral with the Trapezoidal Rule and $n = 4$.

(b) Approximate the definite integral with Simpson's Rule and $n = 4$.

(c) Find the exact value of the integral.

4. $\int_{-1}^{1} x^2 \, dx$

5. $\int_{0}^{10} 5x \, dx$

6. $\int_{0}^{\pi} \sin x \, dx$

7. $\int_{0}^{4} \sqrt{x} \, dx$

8. $\int_{0}^{3} (x^3 + 2x^2 - 5x + 7) \, dx$

9. $\int_{0}^{1} x^4 \, dx$

10. $\int_{0}^{2\pi} \cos x \, dx$

11. $\int_{-3}^{3} \sqrt{9 - x^2} \, dx$

In Exercises 12 – 19, approximate the definite integral with the Trapezoidal Rule and Simpson's Rule, with $n = 6$.

12. $\int_{0}^{1} \cos\left(x^2\right) \, dx$

13. $\int_{-1}^{1} e^{x^2} \, dx$

14. $\int_{0}^{5} \sqrt{x^2 + 1} \, dx$

15. $\int_{0}^{\pi} x \sin x \, dx$

16. $\int_{0}^{\pi/2} \sqrt{\cos x} \, dx$

17. $\int_{1}^{4} \ln x \, dx$

18. $\int_{-1}^{1} \frac{1}{\sin x + 2} \, dx$

19. $\int_{0}^{6} \frac{1}{\sin x + 2} \, dx$

In Exercises 20 – 23, find n such that the error in approximating the given definite integral is less than 0.0001 when using:

(a) the Trapezoidal Rule

(b) Simpson's Rule

20. $\int_{0}^{\pi} \sin x \, dx$

21. $\int_{1}^{4} \frac{1}{\sqrt{x}} \, dx$

22. $\int_{0}^{\pi} \cos\left(x^2\right) \, dx$

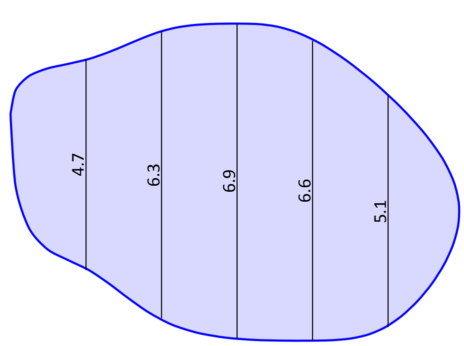
23. $\int_{0}^{5} x^4 \, dx$

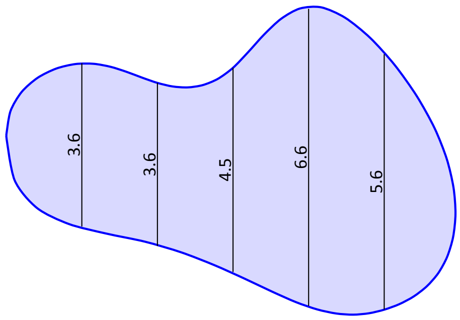
In Exercises 24 – 25, a region is given. Find the area of the region using Simpson's Rule:

(a) where the measurements are in centimeters, taken in 1 cm increments, and

(b) where the measurements are in hundreds of yards, taken in 100 yd increments.

24.



25. 

3: Applications of Integration

We begin this chapter with a reminder of a few key concepts from Chapter 1. Let f be a continuous function on $[a,b]$ which is partitioned into n equally spaced subintervals as

$$a < x_1 < x_2 < \cdots < x_n < x_{n+1} = b.$$

Let $\Delta x = (b-a)/n$ denote the length of the subintervals, and let c_i be any x-value in the i^{th} subinterval. Definition 1.3.1 states that the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x$$

is a *Riemann Sum.* Riemann Sums are often used to approximate some quantity (area, volume, work, pressure, etc.). The *approximation* becomes *exact* by taking the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x.$$

Theorem 1.3.2 connects limits of Riemann Sums to definite integrals:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)\ dx.$$

Finally, the Fundamental Theorem of Calculus states how definite integrals can be evaluated using antiderivatives.

This chapter employs the following technique to a variety of applications. Suppose the value Q of a quantity is to be calculated. We first approximate the value of Q using a Riemann Sum, then find the exact value via a definite integral. We spell out this technique in the following Key Idea.

Key Idea 3.0.1 Application of Definite Integrals Strategy

Let a quantity be given whose value Q is to be computed.

1. Divide the quantity into n smaller "subquantities" of value Q_i.
2. Identify a variable x and function $f(x)$ such that each subquantity can be approximated with the product $f(c_i)\Delta x$, where Δx represents a small change in x. Thus $Q_i \approx f(c_i)\Delta x$. A sample approximation $f(c_i)\Delta x$ of Q_i is called a *differential element.*
3. Recognize that $Q = \sum_{i=1}^{n} Q_i \approx \sum_{i=1}^{n} f(c_i)\Delta x$, which is a Riemann Sum.
4. Taking the appropriate limit gives $Q = \int_a^b f(x)\ dx$

This Key Idea will make more sense after we have had a chance to use it several times. We begin with Area Between Curves, which we addressed briefly in Section 1.1.4.

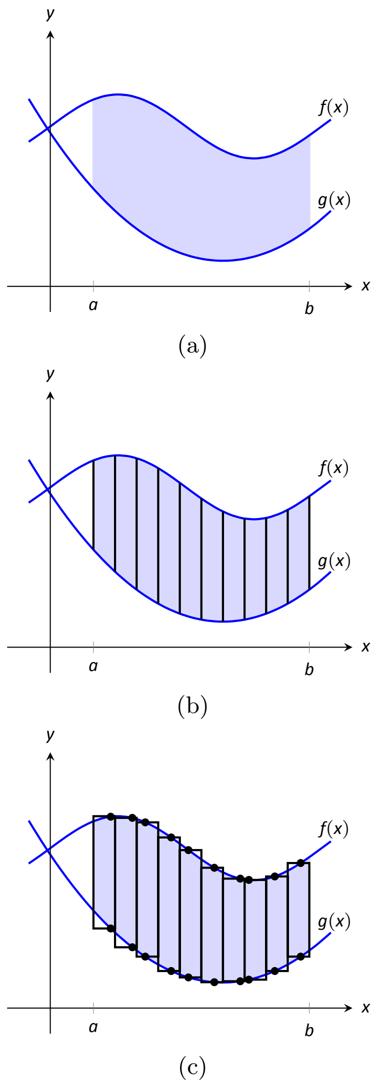


Figure 3.1: Subdividing a region into vertical slices and approximating the areas with rectangles.

3.1 Area Between Curves

We are often interested in knowing the area of a region. Forget momentarily that we addressed this already in Section 1.1.4 and approach it instead using the technique described in Key Idea 3.0.1.

Let Q be the area of a region bounded by continuous functions f and g. If we break the region into many subregions, we have an obvious equation:

Total Area = sum of the areas of the subregions.

The issue to address next is how to systematically break a region into subregions. A graph will help. Consider Figure 3.1 (a) where a region between two curves is shaded. While there are many ways to break this into subregions, one particularly efficient way is to "slice" it vertically, as shown in Figure 3.1 (b), into n equally spaced slices.

We now approximate the area of a slice. Again, we have many options, but using a rectangle seems simplest. Picking any x-value c_i in the i th slice, we set the height of the rectangle to be $f(c_i)-g(c_i)$, the difference of the corresponding y-values. The width of the rectangle is a small difference in x-values, which we represent with Δx. Figure 3.1 (c) shows sample points c_i chosen in each subinterval and appropriate rectangles drawn. (Each of these rectangles represents a differential element.) Each slice has an area approximately equal to $\big(f(c_i)-g(c_i)\big)\Delta x$; hence, the total area is approximately the Riemann Sum

$$Q=\sum_{i=1}^{n}\big(f(c_i)-g(c_i)\big)\Delta x.$$

Taking the limit as $n\to\infty$ gives the exact area as $\int_a^b \big(f(x)-g(x)\big)\,dx$.

Theorem 3.1.1 Area Between Curves (restatement of Theorem 1.4.3)

Let $f(x)$ and $g(x)$ be continuous functions defined on $[a,b]$ where $f(x)\geq g(x)$ for all x in $[a,b]$. The area of the region bounded by the curves $y=f(x)$, $y=g(x)$ and the lines $x=a$ and $x=b$ is

$$\int_a^b \big(f(x)-g(x)\big)\,dx.$$

Example 3.1.1 Finding area enclosed by curves

Find the area of the region bounded by $f(x)=\sin x+2$, $g(x)=\frac{1}{2}\cos(2x)-1$, $x=0$ and $x=4\pi$, as shown in Figure 3.2.

Notes:

Solution The graph verifies that the upper boundary of the region is given by f and the lower bound is given by g. Therefore the area of the region is the value of the integral

$$\int_0^{4\pi}\big(f(x)-g(x)\big)\,dx = \int_0^{4\pi}\Big(\sin x + 2 - \big(\tfrac{1}{2}\cos(2x)-1\big)\Big)\,dx$$
$$= -\cos x - \frac{1}{4}\sin(2x) + 3x\Big|_0^{4\pi}$$
$$= 12\pi \approx 37.7\text{ units}^2.$$

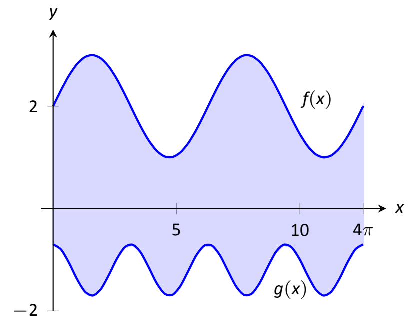


Figure 3.2: Graphing an enclosed region in Example 3.1.1.

Example 3.1.2 Finding total area enclosed by curves
Find the total area of the region enclosed by the functions $f(x) = -2x+5$ and $g(x) = x^3 - 7x^2 + 12x - 3$ as shown in Figure 3.3.

Solution A quick calculation shows that $f = g$ at $x = 1, 2$ and 4. One can proceed thoughtlessly by computing $\int_1^4 \big(f(x)-g(x)\big)\,dx$, but this ignores the fact that on $[1,2]$, $g(x) > f(x)$. (In fact, the thoughtless integration returns $-9/4$, hardly the expected value of an *area*.) Thus we compute the total area by breaking the interval $[1,4]$ into two subintervals, $[1,2]$ and $[2,4]$ and using the proper integrand in each.

$$\text{Total Area} = \int_1^2 \big(g(x)-f(x)\big)\,dx + \int_2^4 \big(f(x)-g(x)\big)\,dx$$
$$= \int_1^2 \big(x^3 - 7x^2 + 14x - 8\big)\,dx + \int_2^4 \big(-x^3 + 7x^2 - 14x + 8\big)\,dx$$
$$= 5/12 + 8/3$$
$$= 37/12 = 3.083\text{ units}^2.$$

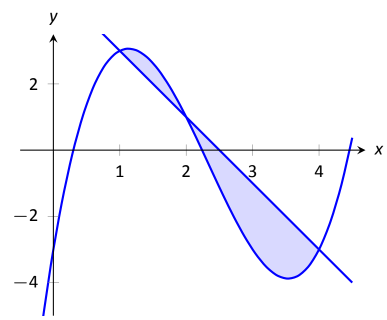


Figure 3.3: Graphing a region enclosed by two functions in Example 3.1.2.

The previous example makes note that we are expecting area to be *positive*. When first learning about the definite integral, we interpreted it as "signed area under the curve," allowing for "negative area." That doesn't apply here; area is to be positive.

The previous example also demonstrates that we often have to break a given region into subregions before applying Theorem 3.1.1. The following example shows another situation where this is applicable, along with an alternate view of applying the Theorem.

Example 3.1.3 Finding area: integrating with respect to y
Find the area of the region enclosed by the functions $y = \sqrt{x} + 2$, $y = -(x-1)^2 + 3$ and $y = 2$, as shown in Figure 3.4.

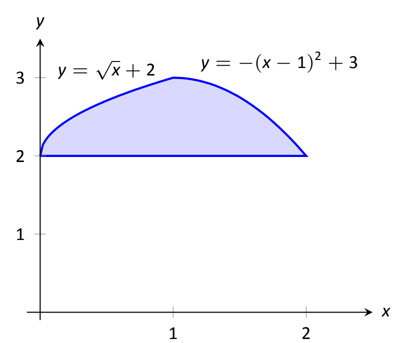


Figure 3.4: Graphing a region for Example 3.1.3.

Notes:

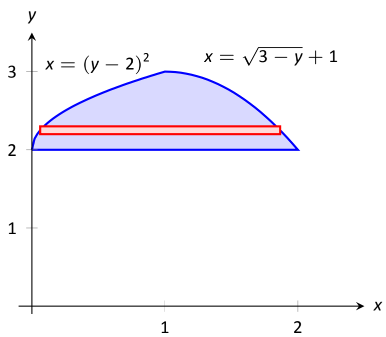


Figure 3.5: The region used in Example 3.1.3 with boundaries relabeled as functions of y.

Solution We give two approaches to this problem. In the first approach, we notice that the region's "top" is defined by two different curves. On $[0,1]$, the top function is $y = \sqrt{x}+2$; on $[1,2]$, the top function is $y = -(x-1)^2+3$. Thus we compute the area as the sum of two integrals:

$$\begin{aligned}\text{Total Area} &= \int_0^1 \Big(\big(\sqrt{x}+2\big)-2\Big)\,dx + \int_1^2 \Big(\big(-(x-1)^2+3\big)-2\Big)\,dx\\ &= 2/3+2/3\\ &= 4/3.\end{aligned}$$

The second approach is clever and very useful in certain situations. We are used to viewing curves as functions of x; we input an x-value and a y-value is returned. Some curves can also be described as functions of y: input a y-value and an x-value is returned. We can rewrite the equations describing the boundary by solving for x:

$$y = \sqrt{x}+2 \quad\Rightarrow\quad x = (y-2)^2$$

$$y = -(x-1)^2+3 \quad\Rightarrow\quad x = \sqrt{3-y}+1.$$

Figure 3.5 shows the region with the boundaries relabeled. A differential element, a horizontal rectangle, is also pictured. The width of the rectangle is a small change in y: Δy. The height of the rectangle is a difference in x-values. The "top" x-value is the largest value, i.e., the rightmost. The "bottom" x-value is the smaller, i.e., the leftmost. Therefore the height of the rectangle is

$$\big(\sqrt{3-y}+1\big)-(y-2)^2.$$

The area is found by integrating the above function with respect to y with the appropriate bounds. We determine these by considering the y-values the region occupies. It is bounded below by $y=2$, and bounded above by $y=3$. That is, both the "top" and "bottom" functions exist on the y interval $[2,3]$. Thus

$$\begin{aligned}\text{Total Area} &= \int_2^3 \big(\sqrt{3-y}+1-(y-2)^2\big)\,dy\\ &= \Big(-\frac{2}{3}(3-y)^{3/2}+y-\frac{1}{3}(y-2)^3\Big)\Big|_2^3\\ &= 4/3.\end{aligned}$$

Notes:

This calculus–based technique of finding area can be useful even with shapes that we normally think of as "easy." Example 3.1.4 computes the area of a triangle. While the formula "$\frac{1}{2} \times$ base $\times$ height" is well known, in arbitrary triangles it can be nontrivial to compute the height. Calculus makes the problem simple.

Example 3.1.4 Finding the area of a triangle

Compute the area of the regions bounded by the lines
$y = x + 1$, $y = -2x + 7$ and $y = -\frac{1}{2}x + \frac{5}{2}$, as shown in Figure 3.6.

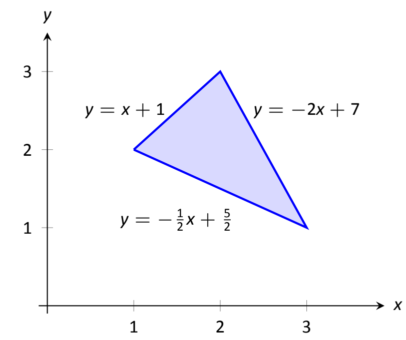


Figure 3.6: Graphing a triangular region in Example 3.1.4.

Solution Recognize that there are two "top" functions to this region, causing us to use two definite integrals.

$$\begin{aligned}\text{Total Area} &= \int_1^2 \big((x+1) - (-\frac{1}{2}x + \frac{5}{2})\big)\ dx + \int_2^3 \big((-2x+7) - (-\frac{1}{2}x + \frac{5}{2})\big)\ dx \\ &= 3/4 + 3/4 \\ &= 3/2.\end{aligned}$$

We can also approach this by converting each function into a function of y. This also requires 2 integrals, so there isn't really any advantage to doing so. We do it here for demonstration purposes.

The "top" function is always $x = \frac{7-y}{2}$ while there are two "bottom" functions. Being mindful of the proper integration bounds, we have

$$\begin{aligned}\text{Total Area} &= \int_1^2 \big(\frac{7-y}{2} - (5-2y)\big)\ dy + \int_2^3 \big(\frac{7-y}{2} - (y-1)\big)\ dy \\ &= 3/4 + 3/4 \\ &= 3/2.\end{aligned}$$

Of course, the final answer is the same. (It is interesting to note that the area of all 4 subregions used is 3/4. This is coincidental.)

While we have focused on producing exact answers, we are also able to make approximations using the principle of Theorem 3.1.1. The integrand in the theorem is a distance ("top minus bottom"); integrating this distance function gives an area. By taking discrete measurements of distance, we can approximate an area using numerical integration techniques developed in Section 2.7. The following example demonstrates this.

Example 3.1.5 Numerically approximating area

To approximate the area of a lake, shown in Figure 3.7 (a), the "length" of the lake is measured at 200-foot increments as shown in Figure 3.7 (b),

Notes:

where the lengths are given in hundreds of feet. Approximate the area of the lake.

Solution The measurements of length can be viewed as measuring "top minus bottom" of two functions. The exact answer is found by integrating $\int_0^{12} \big(f(x)-g(x)\big)\,dx$, but of course we don't know the functions f and g. Our discrete measurements instead allow us to approximate.

We have the following data points:

$$(0,0),\ (2,2.25),\ (4,5.08),\ (6,6.35),\ (8,5.21),\ (10,2.76),\ (12,0).$$

We also have that $\Delta x = \frac{b-a}{n} = 2$, so Simpson's Rule gives

$$\begin{aligned}\text{Area} &\approx \frac{2}{3}\Big(1\cdot 0 + 4\cdot 2.25 + 2\cdot 5.08 + 4\cdot 6.35 + 2\cdot 5.21 + 4\cdot 2.76 + 1\cdot 0\Big)\\ &= 44.01\overline{3}\ \text{units}^2.\end{aligned}$$

Since the measurements are in hundreds of feet, $\text{units}^2 = (100\ \text{ft})^2 = 10,000\ \text{ft}^2$, giving a total area of $440,133\ \text{ft}^2$. (Since we are approximating, we'd likely say the area was about $440,000\ \text{ft}^2$, which is a little more than 10 acres.)

In the next section we apply our applications–of–integration techniques to finding the volumes of certain solids.

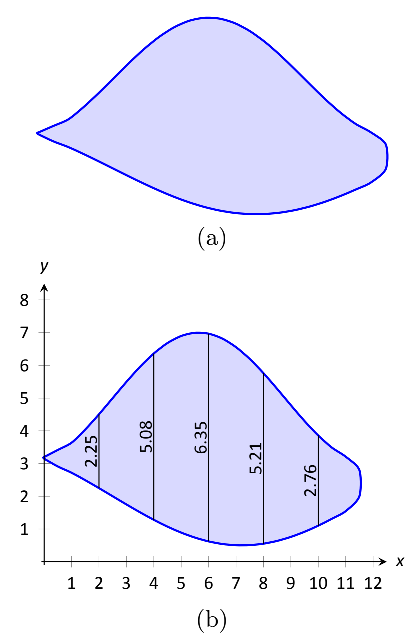


Figure 3.7: (a) A sketch of a lake, and (b) the lake with length measurements.

Notes:

Exercises 3.1

Terms and Concepts

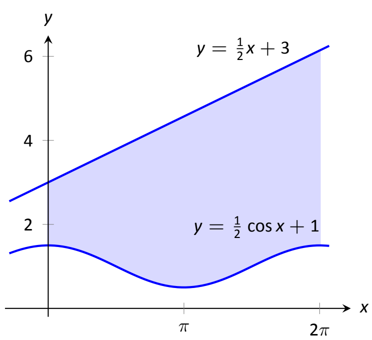
1. T/F: The area between curves is always positive.

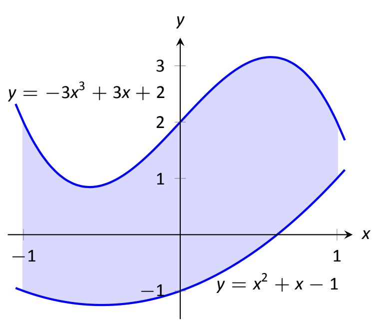
2. T/F: Calculus can be used to find the area of basic geometric shapes.

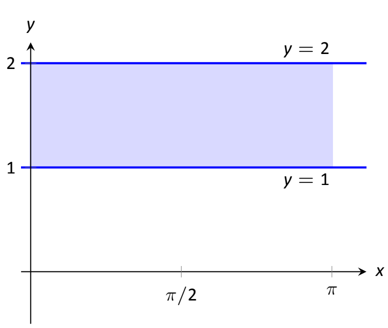
3. In your own words, describe how to find the total area enclosed by $y = f(x)$ and $y = g(x)$.

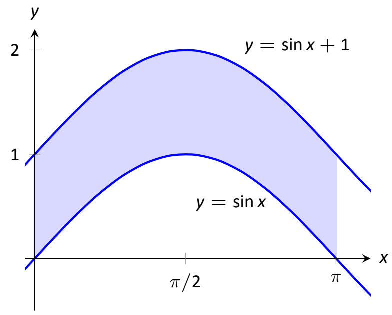
Problems

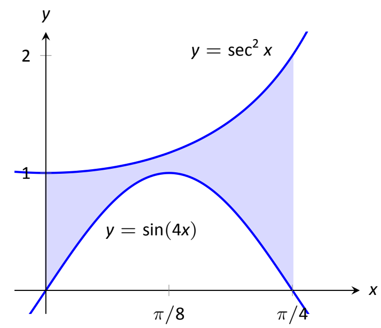
In Exercises 4 – 10, find the area of the shaded region in the given graph.

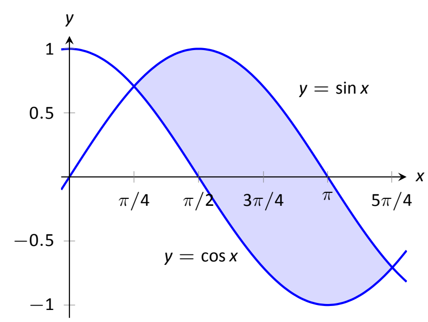
4. 

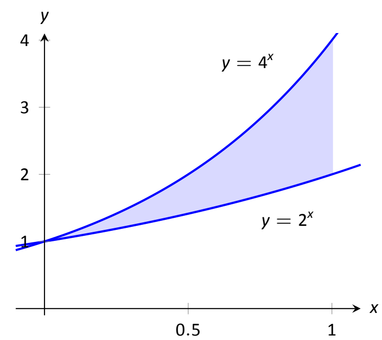
5. 

6. 

7. 

8. 

9. 

10. 

In Exercises 11 – 16, find the total area enclosed by the functions f and g.

11. $f(x) = 2x^2 + 5x - 3$, $g(x) = x^2 + 4x - 1$

12. $f(x) = x^2 - 3x + 2$, $g(x) = -3x + 3$

13. $f(x) = \sin x$, $g(x) = 2x/\pi$

14. $f(x) = x^3 - 4x^2 + x - 1$, $g(x) = -x^2 + 2x - 4$

15. $f(x) = x$, $g(x) = \sqrt{x}$

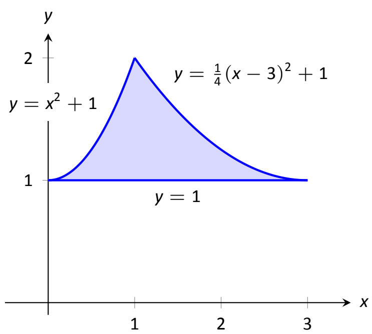
16. $f(x) = -x^3 + 5x^2 + 2x + 1$, $g(x) = 3x^2 + x + 3$

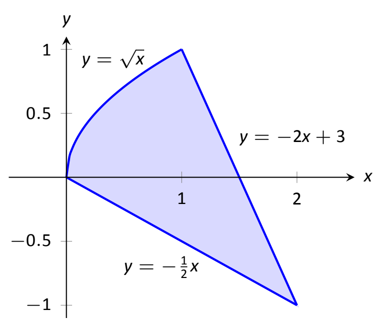
17. The functions $f(x) = \cos(2x)$ and $g(x) = \sin x$ intersect infinitely many times, forming an infinite number of repeated, enclosed regions. Find the areas of these regions.

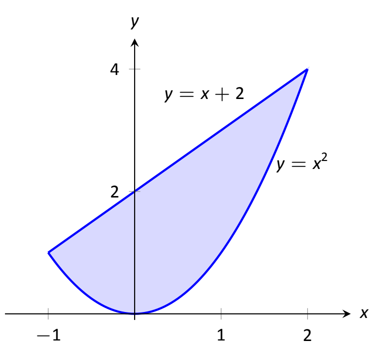
In Exercises 18 – 22, find the area of the enclosed region in two ways:

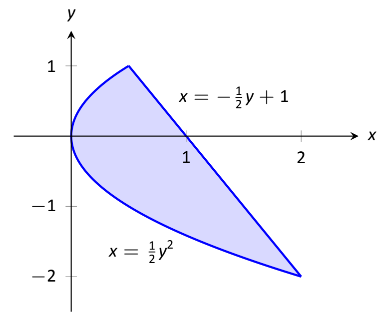
1. **by treating the boundaries as functions of x, and**

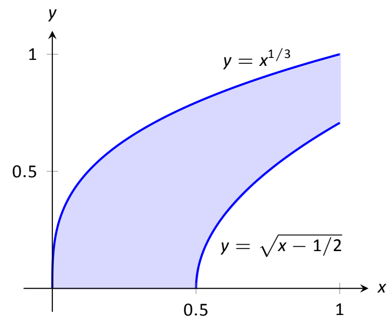
2. **by treating the boundaries as functions of y.**

18. 

19. 

20. 

21. 

22. 

In Exercises 23 – 26, find the area triangle formed by the given three points.

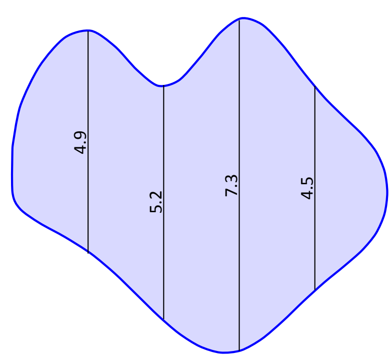
23. $(1, 1)$, $(2, 3)$, and $(3, 3)$

24. $(-1, 1)$, $(1, 3)$, and $(2, -1)$

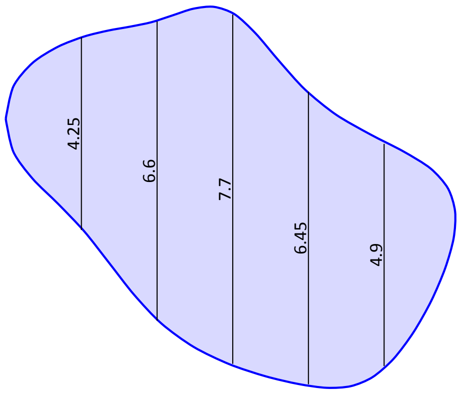
25. $(1, 1)$, $(3, 3)$, and $(3, 3)$

26. $(0, 0)$, $(2, 5)$, and $(5, 2)$

27. Use the Trapezoidal Rule to approximate the area of the pictured lake whose lengths, in hundreds of feet, are measured in 100-foot increments.



28. Use Simpson's Rule to approximate the area of the pictured lake whose lengths, in hundreds of feet, are measured in 200-foot increments.


4.25
6.6
7.7
6.45
4.9

3.2 Volume by Cross-Sectional Area; Disk and Washer Methods

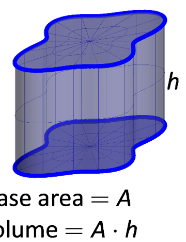


Figure 3.8: The volume of a general right cylinder

The volume of a general right cylinder, as shown in Figure 3.8, is

Area of the base × height.

We can use this fact as the building block in finding volumes of a variety of shapes.

Given an arbitrary solid, we can *approximate* its volume by cutting it into n thin slices. When the slices are thin, each slice can be approximated well by a general right cylinder. Thus the volume of each slice is approximately its cross-sectional area × thickness. (These slices are the differential elements.)

By orienting a solid along the x-axis, we can let $A(x_i)$ represent the cross-sectional area of the i^{th} slice, and let Δx_i represent the thickness of this slice (the thickness is a small change in x). The total volume of the solid is approximately:

$$\begin{aligned}\text{Volume} &\approx \sum_{i=1}^{n}\Big[\text{Area} \;\times\; \text{thickness}\Big]\\ &= \sum_{i=1}^{n} A(x_i)\Delta x_i.\end{aligned}$$

Recognize that this is a Riemann Sum. By taking a limit (as the thickness of the slices goes to 0) we can find the volume exactly.

Theorem 3.2.1 Volume By Cross-Sectional Area

The volume V of a solid, oriented along the x-axis with cross-sectional area $A(x)$ from $x=a$ to $x=b$, is

$$V = \int_a^b A(x)\ dx.$$

Example 3.2.1 Finding the volume of a solid

Find the volume of a pyramid with a square base of side length 10 in and a height of 5 in.

Solution There are many ways to "orient" the pyramid along the x-axis; Figure 3.9 gives one such way, with the pointed top of the pyramid at the origin and the x-axis going through the center of the base.

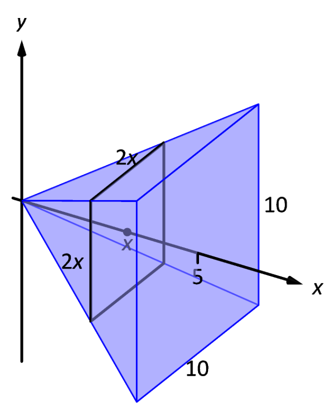


Figure 3.9: Orienting a pyramid along the x-axis in Example 3.2.1.

Notes:

Each cross section of the pyramid is a square; this is a sample differential element. To determine its area $A(x)$, we need to determine the side lengths of the square.

When $x = 5$, the square has side length 10; when $x = 0$, the square has side length 0. Since the edges of the pyramid are lines, it is easy to figure that each cross-sectional square has side length $2x$, giving $A(x) = (2x)^2 = 4x^2$.

If one were to cut a slice out of the pyramid at $x = 3$, as shown in Figure 3.10, one would have a shape with square bottom and top with sloped sides. If the slice were thin, both the bottom and top squares would have sides lengths of about 6, and thus the cross–sectional area of the bottom and top would be about 36in^2. Letting Δx_i represent the thickness of the slice, the volume of this slice would then be about $36\Delta x_i\text{in}^3$.

Cutting the pyramid into n slices divides the total volume into n equally–spaced smaller pieces, each with volume $(2x_i)^2\Delta x$, where x_i is the approximate location of the slice along the x-axis and Δx represents the thickness of each slice. One can approximate total volume of the pyramid by summing up the volumes of these slices:

$$\text{Approximate volume } = \sum_{i=1}^{n}(2x_i)^2\Delta x.$$

Taking the limit as $n \to \infty$ gives the actual volume of the pyramid; recoginizing this sum as a Riemann Sum allows us to find the exact answer using a definite integral, matching the definite integral given by Theorem 3.2.1.

We have

$$\begin{aligned} V &= \lim_{n\to\infty}\sum_{i=1}^{n}(2x_i)^2\Delta x \\ &= \int_0^5 4x^2\,dx \\ &= \frac{4}{3}x^3\Big|_0^5 \\ &= \frac{500}{3}\text{ in}^3 \approx 166.67\text{ in}^3. \end{aligned}$$

We can check our work by consulting the general equation for the volume of a pyramid (see the back cover under “Volume of A General Cone”):

$$\tfrac{1}{3} \times \text{area of base} \times \text{height}.$$

Certainly, using this formula from geometry is faster than our new method, but the calculus–based method can be applied to much more than just

Notes:

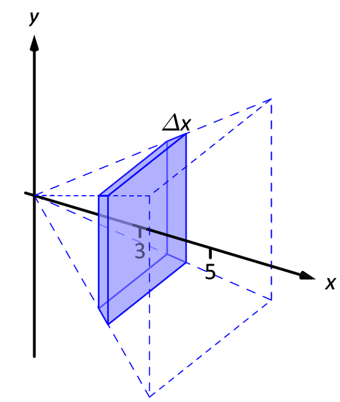


Figure 3.10: Cutting a slice in they pyramid in Example 3.2.1 at $x = 3$.

cones.

An important special case of Theorem 3.2.1 is when the solid is a **solid of revolution**, that is, when the solid is formed by rotating a shape around an axis.

Start with a function $y = f(x)$ from $x = a$ to $x = b$. Revolving this curve about a horizontal axis creates a three-dimensional solid whose cross sections are disks (thin circles). Let $R(x)$ represent the radius of the cross-sectional disk at x; the area of this disk is $\pi R(x)^2$. Applying Theorem 3.2.1 gives the Disk Method.

Key Idea 3.2.1 The Disk Method

Let a solid be formed by revolving the curve $y = f(x)$ from $x = a$ to $x = b$ around a horizontal axis, and let $R(x)$ be the radius of the cross-sectional disk at x. The volume of the solid is

$$V = \pi \int_a^b R(x)^2 \, dx.$$

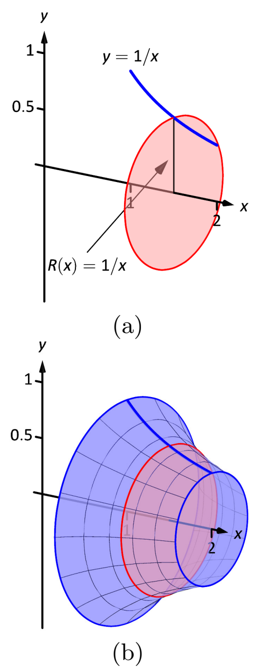


Figure 3.11: Sketching a solid in Example 3.2.2.

Example 3.2.2 Finding volume using the Disk Method

Find the volume of the solid formed by revolving the curve $y = 1/x$, from $x = 1$ to $x = 2$, around the x-axis.

Solution A sketch can help us understand this problem. In Figure 3.11(a) the curve $y = 1/x$ is sketched along with the differential element – a disk – at x with radius $R(x) = 1/x$. In Figure 3.11 (b) the whole solid is pictured, along with the differential element.

The volume of the differential element shown in part (a) of the figure is approximately $\pi R(x_i)^2 \Delta x$, where $R(x_i)$ is the radius of the disk shown and Δx is the thickness of that slice. The radius $R(x_i)$ is the distance from the x-axis to the curve, hence $R(x_i) = 1/x_i$.

Slicing the solid into n equally–spaced slices, we can approximate the total volume by adding up the approximate volume of each slice:

$$\text{Approximate volume } = \sum_{i=1}^{n} \pi \left(\frac{1}{x_i}\right)^2 \Delta x.$$

Taking the limit of the above sum as $n \to \infty$ gives the actual volume; recognizing this sum as a Riemann sum allows us to evaluate the limit with a definite integral, which matches the formula given in Key Idea 3.2.1:

Notes:

$$\begin{aligned} V &= \lim_{n\to\infty}\sum_{i=1}^{n}\pi\left(\frac{1}{x_i}\right)^2\Delta x \\ &= \pi\int_1^2\left(\frac{1}{x}\right)^2\,dx \\ &= \pi\int_1^2\frac{1}{x^2}\,dx \\ &= \pi\left[-\frac{1}{x}\right]\Bigg|_1^2 \\ &= \pi\left[-\frac{1}{2}-(-1)\right] \\ &= \frac{\pi}{2}\text{ units}^3. \end{aligned}$$

While Key Idea 3.2.1 is given in terms of functions of x, the principle involved can be applied to functions of y when the axis of rotation is vertical, not horizontal. We demonstrate this in the next example.

Example 3.2.3 Finding volume using the Disk Method

Find the volume of the solid formed by revolving the curve $y = 1/x$, from $x = 1$ to $x = 2$, about the y-axis.

Solution Since the axis of rotation is vertical, we need to convert the function into a function of y and convert the x-bounds to y-bounds. Since $y = 1/x$ defines the curve, we rewrite it as $x = 1/y$. The bound $x = 1$ corresponds to the y-bound $y = 1$, and the bound $x = 2$ corresponds to the y-bound $y = 1/2$.

Thus we are rotating the curve $x = 1/y$, from $y = 1/2$ to $y = 1$ about the y-axis to form a solid. The curve and sample differential element are sketched in Figure 3.12 (a), with a full sketch of the solid in Figure 3.12 (b). We integrate to find the volume:

$$\begin{aligned} V &= \pi\int_{1/2}^{1}\frac{1}{y^2}\,dy \\ &= -\frac{\pi}{y}\Bigg|_{1/2}^{1} \\ &= \pi\text{ units}^3. \end{aligned}$$

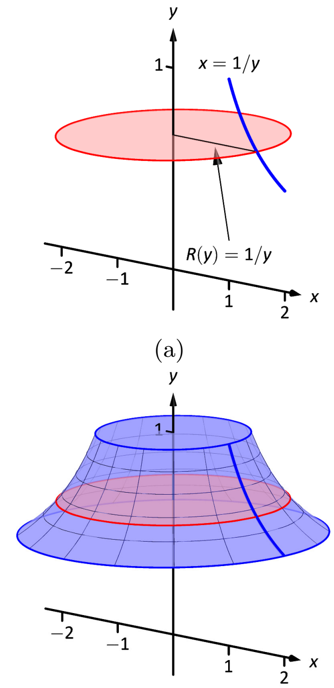


Figure 3.12: Sketching a solid in Example 3.2.3.

Notes:

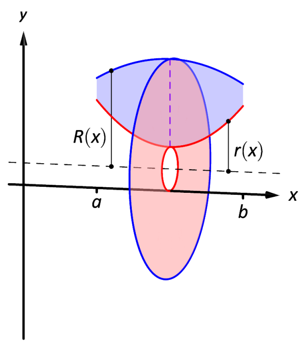


Figure 3.14: Establishing the Washer Method; see also Figure 3.13.

We can also compute the volume of solids of revolution that have a hole in the center. The general principle is simple: compute the volume of the solid irrespective of the hole, then subtract the volume of the hole. If the outside radius of the solid is $R(x)$ and the inside radius (defining the hole) is $r(x)$, then the volume is

$$V = \pi \int_a^b R(x)^2 \, dx - \pi \int_a^b r(x)^2 \, dx = \pi \int_a^b \left(R(x)^2 - r(x)^2\right) \, dx.$$

One can generate a solid of revolution with a hole in the middle by revolving a region about an axis. Consider Figure 3.13(a), where a region is sketched along with a dashed, horizontal axis of rotation. By rotating the region about the axis, a solid is formed as sketched in Figure 3.13(b). The outside of the solid has radius $R(x)$, whereas the inside has radius $r(x)$. Each cross section of this solid will be a washer (a disk with a hole in the center) as sketched in Figure 3.14(c). This leads us to the Washer Method.

Key Idea 3.2.2 The Washer Method

Let a region bounded by $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$ be rotated about a horizontal axis that does not intersect the region, forming a solid. Each cross section at x will be a washer with outside radius $R(x)$ and inside radius $r(x)$. The volume of the solid is

$$V = \pi \int_a^b \left(R(x)^2 - r(x)^2\right) \, dx.$$

Even though we introduced it first, the Disk Method is just a special case of the Washer Method with an inside radius of $r(x) = 0$.

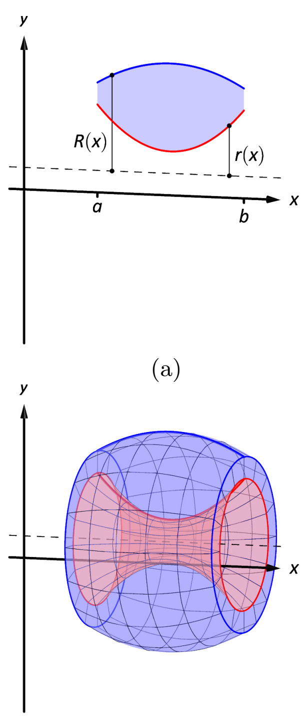


Figure 3.13: Establishing the Washer Method; see also Figure 3.14.

Example 3.2.4 Finding volume with the Washer Method

Find the volume of the solid formed by rotating the region bounded by $y = x^2 - 2x + 2$ and $y = 2x - 1$ about the x-axis.

Solution A sketch of the region will help, as given in Figure 3.15(a). Rotating about the x-axis will produce cross sections in the shape of washers, as shown in Figure 3.15(b); the complete solid is shown in part (c). The outside radius of this washer is $R(x) = 2x + 1$; the inside radius

Notes:

is $r(x) = x^2 - 2x + 2$. As the region is bounded from $x = 1$ to $x = 3$, we integrate as follows to compute the volume.

$$\begin{aligned} V &= \pi \int_1^3 \Big((2x-1)^2 - (x^2-2x+2)^2\Big)\,dx \\ &= \pi \int_1^3 \big(-x^4 + 4x^3 - 4x^2 + 4x - 3\big)\,dx \\ &= \pi \Big[-\frac{1}{5}x^5 + x^4 - \frac{4}{3}x^3 + 2x^2 - 3x\Big]\Big|_1^3 \\ &= \frac{104}{15}\pi \approx 21.78 \text{ units}^3. \end{aligned}$$

When rotating about a vertical axis, the outside and inside radius functions must be functions of y.

Example 3.2.5 Finding volume with the Washer Method

Find the volume of the solid formed by rotating the triangular region with vertices at $(1, 1)$, $(2, 1)$ and $(2, 3)$ about the y-axis.

Solution The triangular region is sketched in Figure 3.16(a); the differential element is sketched in (b) and the full solid is drawn in (c). They help us establish the outside and inside radii. Since the axis of rotation is vertical, each radius is a function of y.

The outside radius $R(y)$ is formed by the line connecting $(2, 1)$ and $(2, 3)$; it is a constant function, as regardless of the y-value the distance from the line to the axis of rotation is 2. Thus $R(y) = 2$.

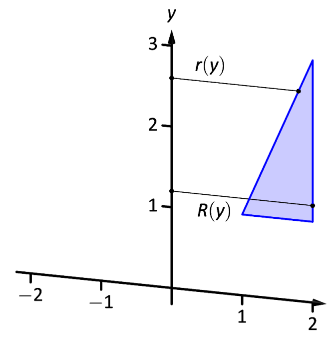
The inside radius is formed by the line connecting $(1, 1)$ and $(2, 3)$. The equation of this line is $y = 2x - 1$, but we need to refer to it as a function of y. Solving for x gives $r(y) = \frac{1}{2}(y + 1)$.

We integrate over the y-bounds of $y = 1$ to $y = 3$. Thus the volume is

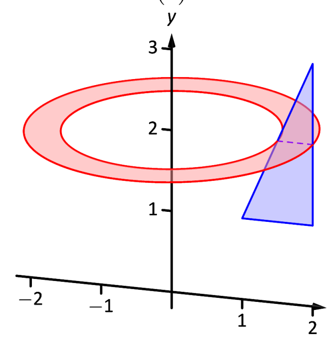
$$\begin{aligned} V &= \pi \int_1^3 \Big(2^2 - \big(\frac{1}{2}(y+1)\big)^2\Big)\,dy \\ &= \pi \int_1^3 \Big(-\frac{1}{4}y^2 - \frac{1}{2}y + \frac{15}{4}\Big)\,dy \\ &= \pi \Big[-\frac{1}{12}y^3 - \frac{1}{4}y^2 + \frac{15}{4}y\Big]\Big|_1^3 \\ &= \frac{10}{3}\pi \approx 10.47 \text{ units}^3. \end{aligned}$$

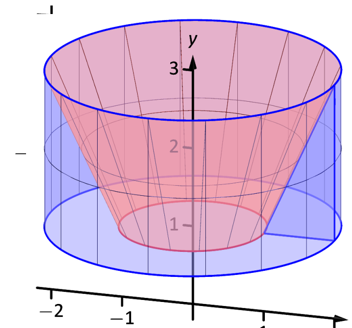
This section introduced a new application of the definite integral. Our default view of the definite integral is that it gives "the area under the

Notes:



(a)





(b)

(c)

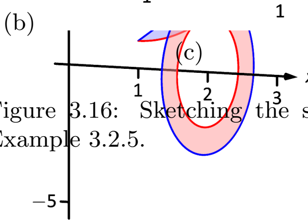
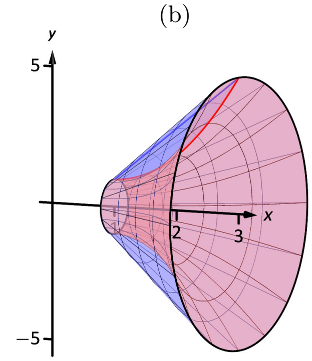


Figure 3.16: Sketching the solid in Example 3.2.5.

(b)



(c)

Figure 3.15: Sketching the differential element and solid in Example 3.2.4.

curve." However, we can establish definite integrals that represent other quantities; in this section, we computed volume.

The ultimate goal of this section is not to compute volumes of solids. That can be useful, but what is more useful is the understanding of this basic principle of integral calculus, outlined in Key Idea 3.0.1: to find the exact value of some quantity,

- we start with an approximation (in this section, slice the solid and approximate the volume of each slice),
- then make the approximation better by refining our original approximation (i.e., use more slices),
- then use limits to establish a definite integral which gives the exact value.

We practice this principle in the next section where we find volumes by slicing solids in a different way.

Notes:

Exercises 3.2

Terms and Concepts

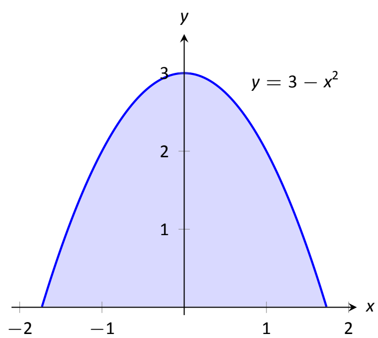
1. T/F: A solid of revolution is formed by revolving a shape around an axis.

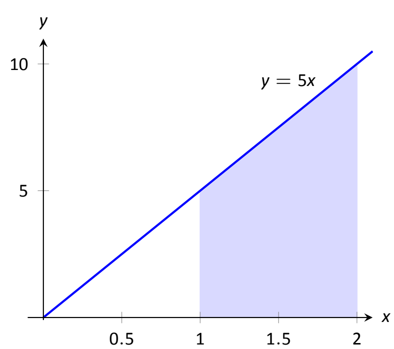
2. In your own words, explain how the Disk and Washer Methods are related.

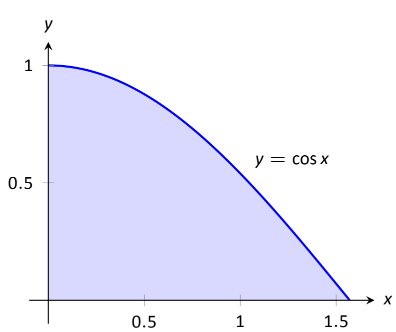
3. Explain the how the units of volume are found in the integral of Theorem 3.2.1: if $A(x)$ has units of in^2, how does $\int A(x)\,dx$ have units of in^3?

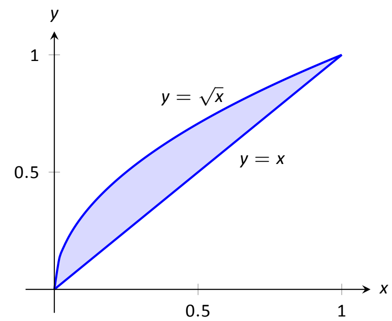
Problems

In Exercises 4 – 7, a region of the Cartesian plane is shaded. Use the Disk/Washer Method to find the volume of the solid of revolution formed by revolving the region about the x-axis.

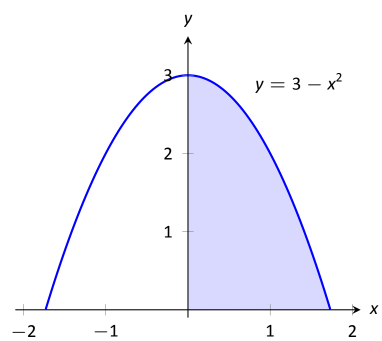
4.


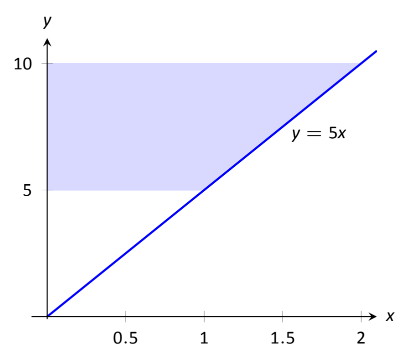
5.


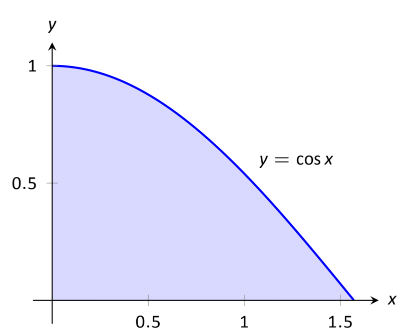
6.


7.


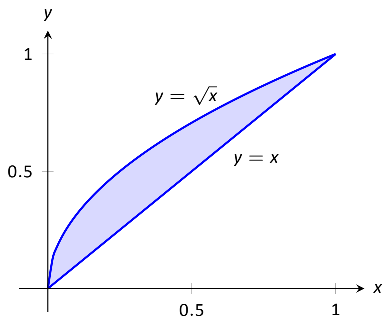
In Exercises 8 – 11, a region of the Cartesian plane is shaded. Use the Disk/Washer Method to find the volume of the solid of revolution formed by revolving the region about the y-axis.

8.


9.


10.


(Hint: Integration By Parts will be necessary, twice. First let $u = \arccos^2 x$, then let $u = \arccos x$.)

11. 

In Exercises 12 – 17, a region of the Cartesian plane is described. Use the Disk/Washer Method to find the volume of the solid of revolution formed by rotating the region about each of the given axes.

12. Region bounded by: $y = \sqrt{x}$, $y = 0$ and $x = 1$.
Rotate about:

    (a) the x-axis
    (b) $y = 1$
    (c) the y-axis
    (d) $x = 1$

13. Region bounded by: $y = 4 - x^2$ and $y = 0$.
Rotate about:

    (a) the x-axis
    (b) $y = 4$
    (c) $y = -1$
    (d) $x = 2$

14. The triangle with vertices $(1, 1)$, $(1, 2)$ and $(2, 1)$.
Rotate about:

    (a) the x-axis
    (b) $y = 2$
    (c) the y-axis
    (d) $x = 1$

15. Region bounded by $y = x^2 - 2x + 2$ and $y = 2x - 1$.
Rotate about:

    (a) the x-axis
    (b) $y = 1$
    (c) $y = 5$

16. Region bounded by $y = 1/\sqrt{x^2 + 1}$, $x = -1$, $x = 1$ and the x-axis.
Rotate about:

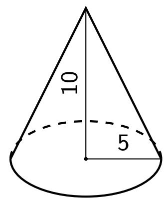
    (a) the x-axis
    (b) $y = 1$
    (c) $y = -1$

17. Region bounded by $y = 2x$, $y = x$ and $x = 2$.
Rotate about:

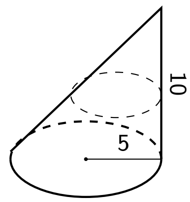
    (a) the x-axis
    (b) $y = 4$
    (c) the y-axis
    (d) $x = 2$

In Exercises 18 – 21, a solid is described. Orient the solid along the x-axis such that a cross-sectional area function $A(x)$ can be obtained, then apply Theorem 3.2.1 to find the volume of the solid.

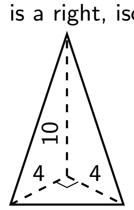
18. A right circular cone with height of 10 and base radius of 5.



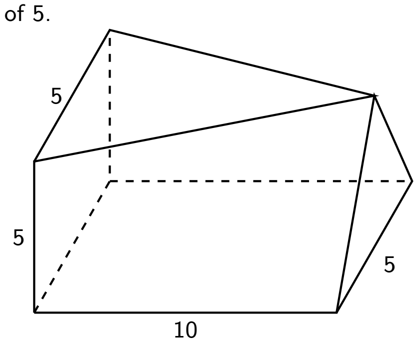
19. A skew right circular cone with height of 10 and base radius of 5. (Hint: all cross-sections are circles.)



20. A right triangular cone with height of 10 and whose base is a right, isosceles triangle with side length 4.



21. A solid with length 10 with a rectangular base and triangular top, wherein one end is a square with side length 5 and the other end is a triangle with base and height of 5.



3.3 The Shell Method

Often a given problem can be solved in more than one way. A particular method may be chosen out of convenience, personal preference, or perhaps necessity. Ultimately, it is good to have options.

The previous section introduced the Disk and Washer Methods, which computed the volume of solids of revolution by integrating the cross–sectional area of the solid. This section develops another method of computing volume, the **Shell Method.** Instead of slicing the solid perpendicular to the axis of rotation creating cross-sections, we now slice it parallel to the axis of rotation, creating "shells."

Consider Figure 3.17, where the region shown in (a) is rotated around the y-axis forming the solid shown in (b). A small slice of the region is drawn in (a), parallel to the axis of rotation. When the region is rotated, this thin slice forms a **cylindrical shell**, as pictured in part (c) of the figure. The previous section approximated a solid with lots of thin disks (or washers); we now approximate a solid with many thin cylindrical shells.

To compute the volume of one shell, first consider the paper label on a soup can with radius r and height h. What is the area of this label? A simple way of determining this is to cut the label and lay it out flat, forming a rectangle with height h and length $2\pi r$. Thus the area is $A = 2\pi rh$; see Figure 3.18 (a).

Do a similar process with a cylindrical shell, with height h, thickness Δx, and approximate radius r. Cutting the shell and laying it flat forms a rectangular solid with length $2\pi r$, height h and depth Δx. Thus the volume is $V \approx 2\pi rh\Delta x$; see Figure 3.18 (b). (We say "approximately" since our radius was an approximation.)

By breaking the solid into n cylindrical shells, we can approximate the volume of the solid as

$$V = \sum_{i=1}^{n} 2\pi r_i h_i \Delta x_i,$$

where r_i, h_i and Δx_i are the radius, height and thickness of the i^{th} shell, respectively.

This is a Riemann Sum. Taking a limit as the thickness of the shells approaches 0 leads to a definite integral.

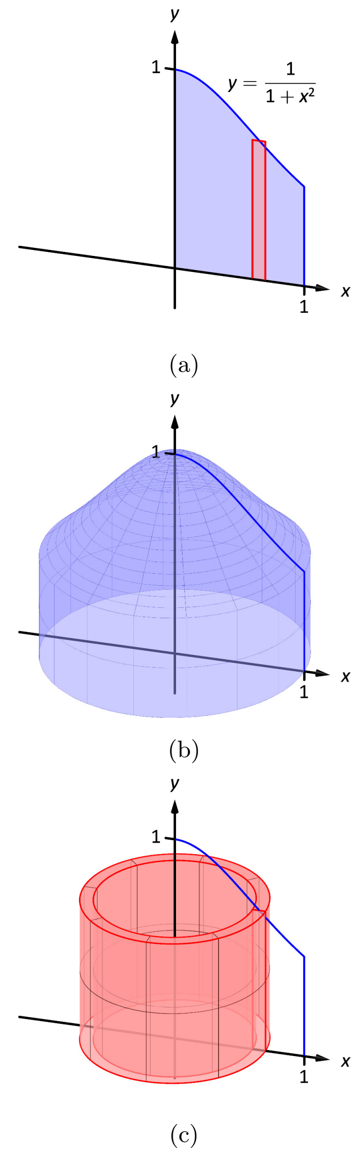


Figure 3.17: Introducing the Shell Method.

Notes:

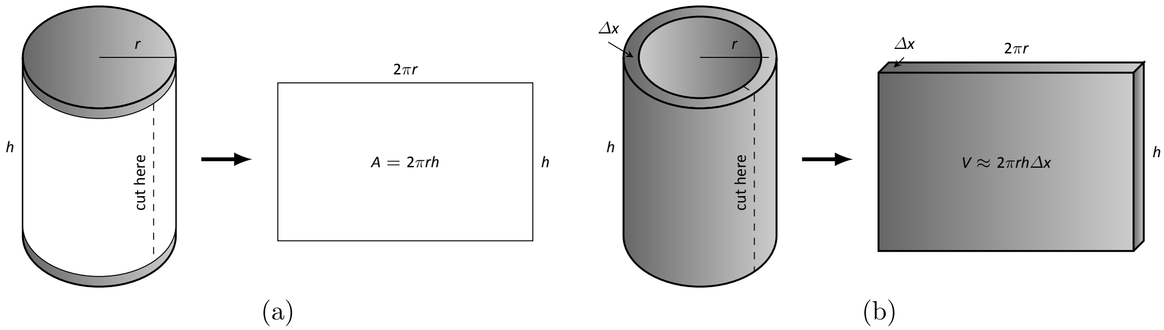


Figure 3.18: Determining the volume of a thin cylindrical shell.

Key Idea 3.3.1 The Shell Method

Let a solid be formed by revolving a region R, bounded by $x = a$ and $x = b$, around a vertical axis. Let $r(x)$ represent the distance from the axis of rotation to x (i.e., the radius of a sample shell) and let $h(x)$ represent the height of the solid at x (i.e., the height of the shell). The volume of the solid is

$$V = 2\pi \int_a^b r(x)h(x)\ dx.$$

Special Cases:

1. When the region R is bounded above by $y = f(x)$ and below by $y = g(x)$, then $h(x) = f(x) - g(x)$.
2. When the axis of rotation is the y-axis (i.e., $x = 0$) then $r(x) = x$.

Let's practice using the Shell Method.

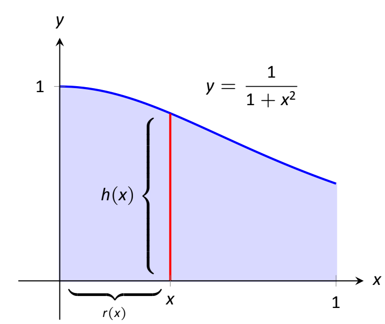


Figure 3.19: Graphing a region in Example 3.3.1.

Example 3.3.1 Finding volume using the Shell Method

Find the volume of the solid formed by rotating the region bounded by $y = 0$, $y = 1/(1 + x^2)$, $x = 0$ and $x = 1$ about the y-axis.

Solution This is the region used to introduce the Shell Method in Figure 3.17, but is sketched again in Figure 3.19 for closer reference. A line is drawn in the region parallel to the axis of rotation representing

Notes:

a shell that will be carved out as the region is rotated about the y-axis. (This is the differential element.)

The distance this line is from the axis of rotation determines $r(x)$; as the distance from x to the y-axis is x, we have $r(x) = x$. The height of this line determines $h(x)$; the top of the line is at $y = 1/(1+x^2)$, whereas the bottom of the line is at $y = 0$. Thus $h(x) = 1/(1+x^2) - 0 = 1/(1+x^2)$. The region is bounded from $x = 0$ to $x = 1$, so the volume is

$$V = 2\pi \int_0^1 \frac{x}{1+x^2}\, dx.$$

This requires substitution. Let $u = 1+x^2$, so $du = 2x\, dx$. We also change the bounds: $u(0) = 1$ and $u(1) = 2$. Thus we have:

$$\begin{aligned} &= \pi \int_1^2 \frac{1}{u}\, du \\ &= \pi \ln u \Big|_1^2 \\ &= \pi \ln 2 \approx 2.178 \text{ units}^3. \end{aligned}$$

Note: in order to find this volume using the Disk Method, two integrals would be needed to account for the regions above and below $y = 1/2$.

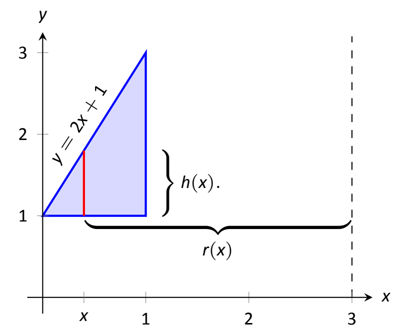
With the Shell Method, nothing special needs to be accounted for to compute the volume of a solid that has a hole in the middle, as demonstrated next.

Example 3.3.2 Finding volume using the Shell Method

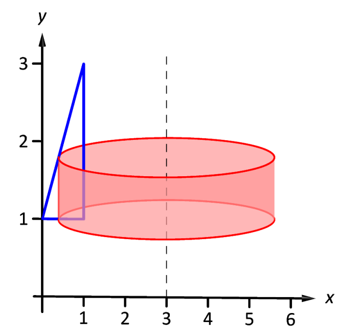
Find the volume of the solid formed by rotating the triangular region determined by the points $(0,1)$, $(1,1)$ and $(1,3)$ about the line $x = 3$.

Solution The region is sketched in Figure 3.20(a) along with the differential element, a line within the region parallel to the axis of rotation. In part (b) of the figure, we see the shell traced out by the differential element, and in part (c) the whole solid is shown.

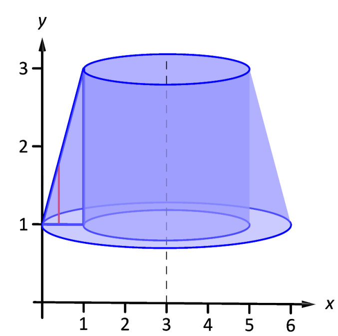
The height of the differential element is the distance from $y = 1$ to $y = 2x+1$, the line that connects the points $(0,1)$ and $(1,3)$. Thus $h(x) = 2x+1-1 = 2x$. The radius of the shell formed by the differential element is the distance from x to $x = 3$; that is, it is $r(x) = 3 - x$. The



(a)



(b)



(c)

Figure 3.20: Graphing a region in Example 3.3.2.

Notes:

x-bounds of the region are $x = 0$ to $x = 1$, giving

$$\begin{aligned} V &= 2\pi \int_0^1 (3-x)(2x)\ dx \\ &= 2\pi \int_0^1 \left(6x - 2x^2\right)\ dx \\ &= 2\pi \left(3x^2 - \frac{2}{3}x^3\right)\Big|_0^1 \\ &= \frac{14}{3}\pi \approx 14.66 \text{ units}^3. \end{aligned}$$

When revolving a region around a horizontal axis, we must consider the radius and height functions in terms of y, not x.

Example 3.3.3 Finding volume using the Shell Method

Find the volume of the solid formed by rotating the region given in Example 3.3.2 about the x-axis.

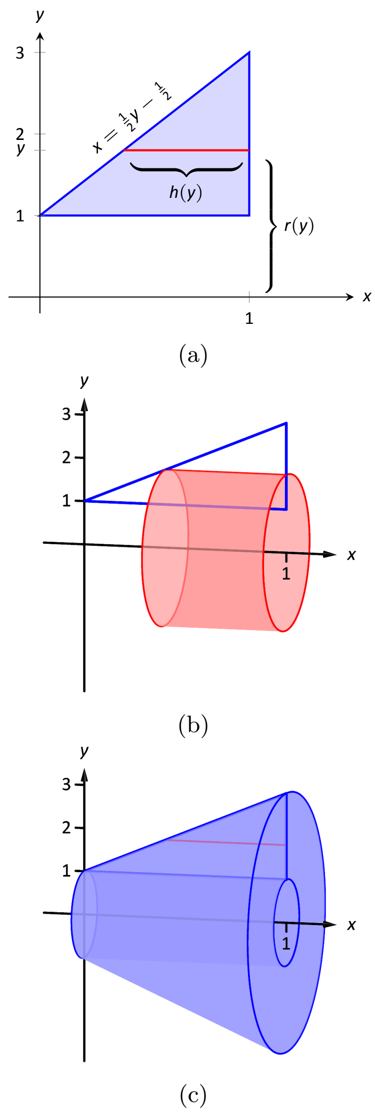


Figure 3.21: Graphing a region in Example 3.3.3.

Solution The region is sketched in Figure 3.21(a) with a sample differential element. In part (b) of the figure the shell formed by the differential element is drawn, and the solid is sketched in (c). (Note that the triangular region looks "short and wide" here, whereas in the previous example the same region looked "tall and narrow." This is because the bounds on the graphs are different.)

The height of the differential element is an x-distance, between $x = \frac{1}{2}y - \frac{1}{2}$ and $x = 1$. Thus $h(y) = 1 - (\frac{1}{2}y - \frac{1}{2}) = -\frac{1}{2}y + \frac{3}{2}$. The radius is the distance from y to the x-axis, so $r(y) = y$. The y bounds of the region are $y = 1$ and $y = 3$, leading to the integral

$$\begin{aligned} V &= 2\pi \int_1^3 \left[y\left(-\frac{1}{2}y + \frac{3}{2}\right)\right]\ dy \\ &= 2\pi \int_1^3 \left[-\frac{1}{2}y^2 + \frac{3}{2}y\right]\ dy \\ &= 2\pi \left[-\frac{1}{6}y^3 + \frac{3}{4}y^2\right]\Big|_1^3 \\ &= 2\pi \left[\frac{9}{4} - \frac{7}{12}\right] \\ &= \frac{10}{3}\pi \approx 10.472 \text{ units}^3. \end{aligned}$$

Notes:

At the beginning of this section it was stated that "it is good to have options." The next example finds the volume of a solid rather easily with the Shell Method, but using the Washer Method would be quite a chore.

Example 3.3.4 Finding volume using the Shell Method
Find the volume of the solid formed by revolving the region bounded by $y = \sin x$ and the x-axis from $x = 0$ to $x = \pi$ about the y-axis.

Solution The region and a differential element, the shell formed by this differential element, and the resulting solid are given in Figure 3.22. The radius of a sample shell is $r(x) = x$; the height of a sample shell is $h(x) = \sin x$, each from $x = 0$ to $x = \pi$. Thus the volume of the solid is

$$V = 2\pi \int_0^\pi x \sin x \; dx.$$

This requires Integration By Parts. Set $u = x$ and $dv = \sin x \; dx$; we leave it to the reader to fill in the rest. We have:

$$\begin{aligned} &= 2\pi \Big[-x\cos x \Big|_0^\pi + \int_0^\pi \cos x \; dx \Big] \\ &= 2\pi \Big[\pi + \sin x \Big|_0^\pi \Big] \\ &= 2\pi \Big[\pi + 0 \Big] \\ &= 2\pi^2 \approx 19.74 \text{ units}^3. \end{aligned}$$

Note that in order to use the Washer Method, we would need to solve $y = \sin x$ for x, requiring the use of the arcsine function. We leave it to the reader to verify that the outside radius function is $R(y) = \pi - \arcsin y$ and the inside radius function is $r(y) = \arcsin y$. Thus the volume can be computed as

$$\pi \int_0^1 \Big[(\pi - \arcsin y)^2 - (\arcsin y)^2 \Big] \; dy.$$

This integral isn't terrible given that the $\arcsin^2 y$ terms cancel, but it is more onerous than the integral created by the Shell Method.

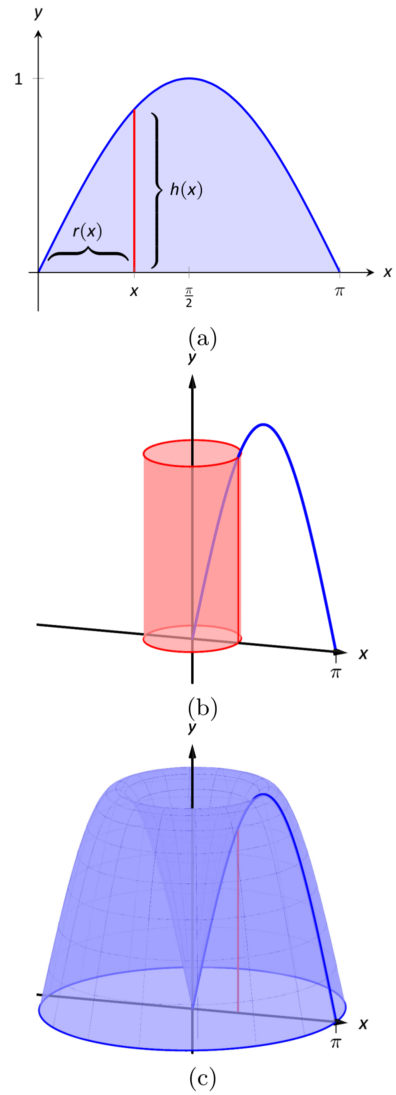


Figure 3.22: Graphing a region in Example 3.3.4.

We end this section with a table summarizing the usage of the Washer and Shell Methods.

Notes:

Key Idea 3.3.2 Summary of the Washer and Shell Methods

Let a region R be given with x-bounds $x = a$ and $x = b$ and y-bounds $y = c$ and $y = d$.

	Washer Method	Shell Method
Horizontal Axis	$\pi \int_a^b \left(R(x)^2 - r(x)^2\right)\, dx$	$2\pi \int_c^d r(y)h(y)\, dy$
Vertical Axis	$\pi \int_c^d \left(R(y)^2 - r(y)^2\right)\, dy$	$2\pi \int_a^b r(x)h(x)\, dx$

As in the previous section, the real goal of this section is not to be able to compute volumes of certain solids. Rather, it is to be able to solve a problem by first approximating, then using limits to refine the approximation to give the exact value. In this section, we approximate the volume of a solid by cutting it into thin cylindrical shells. By summing up the volumes of each shell, we get an approximation of the volume. By taking a limit as the number of equally spaced shells goes to infinity, our summation can be evaluated as a definite integral, giving the exact value.

We use this same principle again in the next section, where we find the length of curves in the plane.

Notes:

Exercises 3.3

Terms and Concepts

1. T/F: A solid of revolution is formed by revolving a shape around an axis.

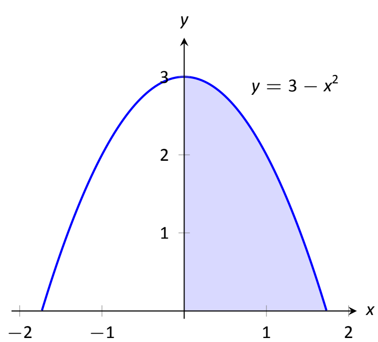
2. T/F: The Shell Method can only be used when the Washer Method fails.

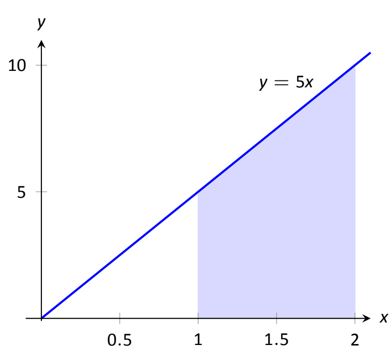
3. T/F: The Shell Method works by integrating cross–sectional areas of a solid.

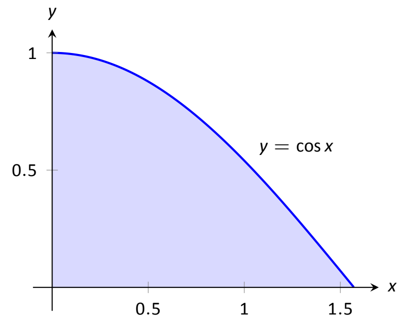
4. T/F: When finding the volume of a solid of revolution that was revolved around a vertical axis, the Shell Method integrates with respect to x.

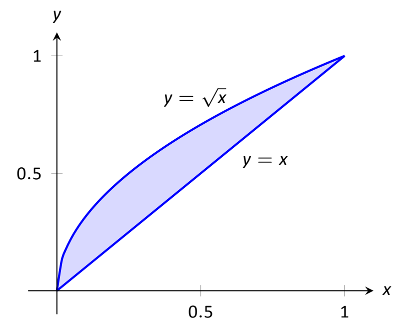
Problems

In Exercises 5 – 8, a region of the Cartesian plane is shaded. Use the Shell Method to find the volume of the solid of revolution formed by revolving the region about the y-axis.

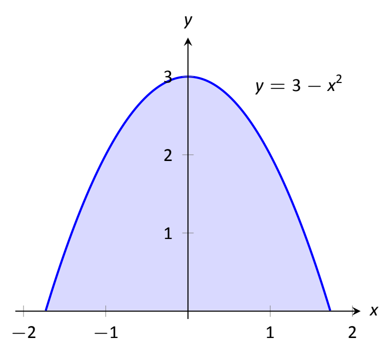
5. 

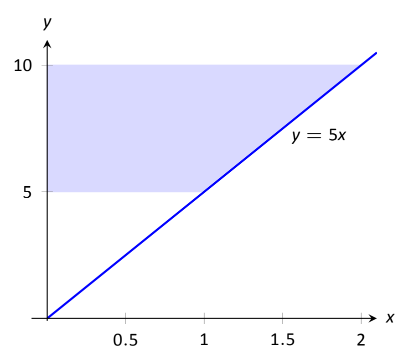
6. 

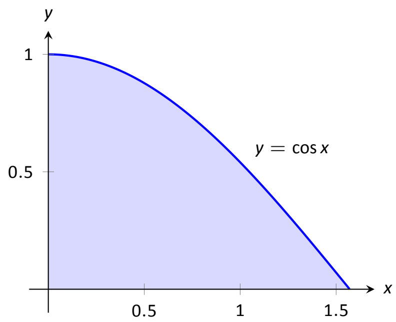
7. 

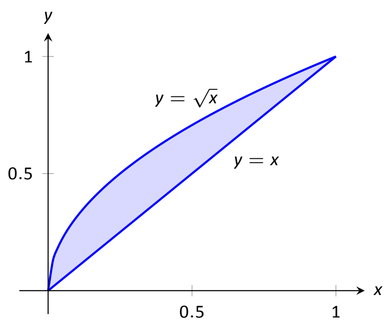
8. 

In Exercises 9 – 12, a region of the Cartesian plane is shaded. Use the Shell Method to find the volume of the solid of revolution formed by revolving the region about the x-axis.

9. 

10. 

11. 

12. 

In Exercises 13 – 18, a region of the Cartesian plane is described. Use the Shell Method to find the volume of the solid of revolution formed by rotating the region about each of the given axes.

13. Region bounded by: $y = \sqrt{x}$, $y = 0$ and $x = 1$.
Rotate about:
   (a) the y-axis
   (b) $x = 1$
   (c) the x-axis
   (d) $y = 1$

14. Region bounded by: $y = 4 - x^2$ and $y = 0$.
Rotate about:
   (a) $x = 2$
   (b) $x = -2$
   (c) the x-axis
   (d) $y = 4$

15. The triangle with vertices $(1, 1)$, $(1, 2)$ and $(2, 1)$.
Rotate about:
   (a) the y-axis
   (b) $x = 1$
   (c) the x-axis
   (d) $y = 2$

16. Region bounded by $y = x^2 - 2x + 2$ and $y = 2x - 1$.
Rotate about:
   (a) the y-axis
   (b) $x = 1$
   (c) $x = -1$

17. Region bounded by $y = 1/\sqrt{x^2 + 1}$, $x = 1$ and the x and y-axes.
Rotate about:
   (a) the y-axis
   (b) $x = 1$

18. Region bounded by $y = 2x$, $y = x$ and $x = 2$.
Rotate about:
   (a) the y-axis
   (b) $x = 2$
   (c) the x-axis
   (d) $y = 4$

4: More Applications

4.1 Continuous Income Streams

(I) Total Income for a Continuous Income Stream:

Let's consider the following example. Suppose that we have a trust that pays us \$2000 a year for 10 years. What is the total amount we will receive from this trust by the end of the $10th$ year? The answer is simple. Since there are ten payments of \$2,000 each, we will receive a total of

$$10 \times \$2,000 = \$20,000.$$

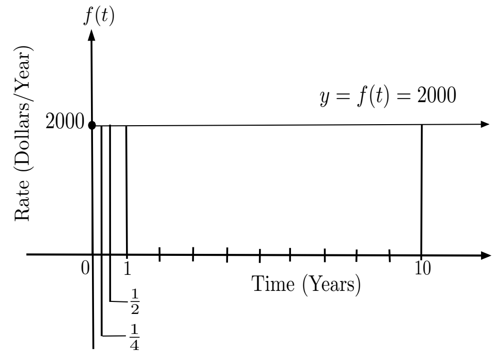


Figure 4.1: The total amount after 10 years

Now, let's look this problem from a calculus point of view. Let's assume that the **income stream** is continuous at a rate of \$2,000 per year. In the Figure 4.1, the area under the curve of $f(t) = 2,000$ from 0 to t represents the income accumulated t years after the start.

For example, for $t = \dfrac{1}{4}$ year, the income would be $\dfrac{1}{4}(2,000) = \$500$. For $t = \dfrac{1}{2}$ year, the income would be $\dfrac{1}{2}(2,000) = \$1,000$. For $t = 1$, the income would be $1(2,000) = \$2,000$. And for $t = 5.3$ years, the income would be $5.3(2000) = \$10,600$. It looks like that these numbers are the areas of rectangles with the same length of 2000 and widths $\dfrac{1}{4}$, $\dfrac{1}{2}$, 1, and 5.3. So, the total income over a ten year period, that is, the area $10(2,000) = \$20,000$. But, this area is the same as the definite integral

$$\int_0^{10} f(t)\, dt = \int_0^{10} 2,000\, dt = 2,000t\Big|_0^{10} = 2,000(10) - 2,000(0) = \$20,000.$$

In general, if the function $f(t)$ is the **rate of flow** (or the **flow rate function**) of a continuous income stream **continuous income stream** (see Figure 4.2), the **total income** produced during the period from $t = a$ to $t = b$ is given by

$$\text{Total income } = \int_a^b f(t)\, dt.$$

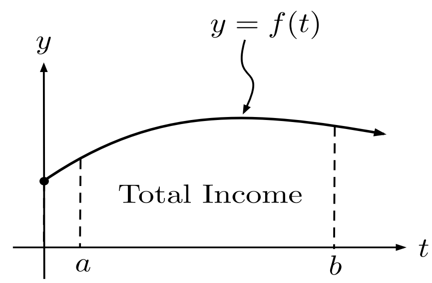


Figure 4.2: The total amount after 10 years

(II) Future Value of a Continuous Income Stream:

Now, we look at the **future value of a continuous income stream**. We have seen in previous calculus class, i.e., our Math 1174 here at Langara, the following continuous compound interest formula

$$A = Pe^{rt} \Rightarrow P = Ae^{-rt},$$

where P is the principal (or present value, which is sometimes denoted by PV), A is the amount (or future value, which is denoted by FV), r is the annual rate of continuous compounding (expressed as a decimal), and t is the time (in years). For example, if money is worth 10% compounded continuously, then the future value of $2,000$ investment in 10 years is

$$A = 2,000e^{0.10\times 10} = \$5,436.56.$$

Now, we want to apply this future value concept to the income produced by continuous income stream. Suppose that $f(t)$ is the rate of flow of a continuous income stream, and the income produced by this continuous income stream is invested as soon as it is received at a rate r, compounded continuously.
We have already known how to compute the **total income** (or **total value**) produced after T years, that is,

$$TV = \int_0^T f(t)\,dt.$$

But, how can we find the total of the income produced and the interest earned by this income TV?

To answer this, we divide the time interval $[0, T]$ into n equal sub-intervals each of the length $\Delta t = \frac{T}{n}$. Let $t_0 = 0,\ t_1 = \frac{T}{n},\ t_2 = \frac{2T}{n},\ \cdots, t_n = \frac{nT}{n} = T$ be the endpoints of the sub-intervals. Then, the amount of money flows into the account on the first interval $[0, t_1]$ is $f(t_0)\Delta t$. This amount remains there for $T - t_1$ years and its future value at the end of the period will $f(t_0)\Delta t \cdot e^{r(T-t_1)}$. Notice that this is just an approximation of the future value because we are working only on time t_1. On $[t_1, t_2]$ we get $f(t_1)\Delta t \cdot e^{r(T-t_2)}$, and so on. We can repeat this for many more sub-intervals. In the last interval, we have

$$f(t_{n-1})\Delta t \cdot e^{r(T-t_n)}.$$

Notes:

Summing all of these up gives

$$f(t_0)\Delta t\cdot e^{r(T-t_1)}+f(t_1)\Delta t\cdot e^{r(T-t_2)}+\cdots+f(t_{n-1})\Delta t\cdot e^{r(T-t_n)}$$

$$=f(t_0)\Delta t{\cdot}e^{r(T-t_0)+r(t_0-t_1)}+f(t_1)\Delta t{\cdot}e^{r(T-t_1)+r(t_1-t_2)}+\cdots+f(t_{n-1})\Delta t{\cdot}e^{r(T-t_{n-1})+r(t_{n-1}-t_n)}$$

$$=\left[f(t_0)\Delta t\cdot e^{r(T-t_0)}e^{r(t_0-t_1)}+f(t_1)\Delta t\cdot e^{r(T-t_1)}e^{r(t_1-t_2)}+\cdots+f(t_{n-1})\Delta t\cdot e^{r(T-t_{n-1})}e^{r(t_{n-1}-t_n)}\right]\Delta t$$

$$=\left[f(t_0)\Delta t\cdot e^{r(T-t_0)}e^{-r\Delta t}+f(t_1)\Delta t\cdot e^{r(T-t_1)}e^{-r\Delta t}+\cdots+f(t_{n-1})\Delta t\cdot e^{r(T-t_{n-1})}e^{-r\Delta t}\right]\Delta t$$

$$=\left[f(t_0)\Delta t\cdot e^{r(T-t_0)}+f(t_1)\Delta t\cdot e^{r(T-t_1)}+\cdots+f(t_{n-1})\Delta t\cdot e^{r(T-t_{n-1})}\right]e^{-r\Delta t}\Delta t=e^{-r\Delta t}\left(\sum_{k=0}^{n-1}f(t_k)e^{r(T-t_k)}\Delta t\right).$$

As n goes to infinity we have $e^{-r\Delta t}=e^{-rT/n}\to 1$. So, we have

$$\lim_{n\to\infty}e^{-r\Delta t}\left(\sum_{k=0}^{n-1}f(t_k)e^{r(T-t_k)}\Delta t\right)=\lim_{n\to\infty}\sum_{k=0}^{n-1}f(t_k)e^{r(T-t_k)}\Delta t=\int_0^T f(t)e^{r(T-t)}\,dt.$$

Therefore, **future value** of a continuous income stream with the flow rate $f(t)$ dollars per year at T years, earning interest at an annual rate r, compounded continuously, is given by

$$FV=\int_0^T f(t)e^{r(T-t)}\,dt.$$

Likewise, we can show that the **present value** of the same income stream is approximately $\sum_{k=0}^{n-1}f(t_k)e^{-rt_k}\Delta t$. Taking the limit as n goes to infinity, gives

$$PV=\int_0^T f(t)e^{-rt}\,dt.$$

Example 4.1.1

Suppose that the rate of change (flow rate) of a continuous income stream produced is given by $f(t)=5,000e^{0.04t}$.

a) Find the total income (TV) during the first 5 years.

b) Find the future value of this income stream at 12%, compounded continuously for 5 years.

c) How much is the total interest earned by this income stream during the five year period?

Notes:

Solution

a) $TV = \int_0^5 f(t)\,dt = \int_0^5 5,000e^{0.04t}\,dt = 125,000e^{0.04t}\Big|_0^5 = \$27,675.$

b) $FV = e^{rT}\int_0^T f(t)e^{-rt}\ dt = e^{0.12\times 5}\int_0^5 5,000e^{0.04t}\cdot e^{-0.12t}\ dt =$ $5,000e^{0.6}\int_0^5 e^{-0.08t}\,dt = 5,000e^{0.6}\dfrac{1}{-0.08}e^{-0.08t}\Big|_0^5 = \$37,545.$

c) $\$37,545 - \$27,675 = \$9,870.$

Example 4.1.2

Suppose you win money for \$1000 per week forever. Assuming that the flow rate is $f(t) = 52,000$ dollars per year and at the interest rate of 4% compounded continuously. What would be the money if you could take it all today?

Solution $PV = \int_0^\infty 52,000e^{-0.04t}\,dt = 52,000\lim_{s\to\infty}\dfrac{1}{-0.04}e^{-0.04t}\Big|_0^s$ $= -1,300,000\lim_{s\to\infty}\left(e^{-0.04s} - 1\right) = 1,300,000$ dollars.

Example 4.1.3

A construction company is working on a project. The cost for this project are: \$1 million today, \$1.5 million in two years time, and after that, \$0.75 million per year continuous over the next three years. Find the present value of the cost if the interest is 3% compounded continuously.

Solution $PV = 1+1.5e^{-0.03\cdot 2}+\int_0^5 0.75\cdot e^{-0.03\cdot t}\,dt = 2.41264680+$ $\int_0^2 0\cdot e^{-0.03\cdot t}\,dt + \int_2^5 0.75\cdot e^{-0.03\cdot t}\,dt = 2.41264680 + \dfrac{0.75}{-0.03}\left[e^{-0.03\cdot 5} - e^{-0.03\cdot 2}\right] = 2.41264680+2.02641393 = 4.43906073$, which is $\$4,439060.73$. Notice that the normal cost would be $1 + 1.5 + 3\times 0.75 = 4.75$ million dollars. So, the company saves money.

Notes:

Example 4.1.4

You want to have $\$200,000$ when you retire. You open an account that pays 8% interest compounded continuously. Each year you put $\$10,000$ into it.

a) How many more years that you have to work before you can retire?

b) However, after you retired, you convert this $\$200,000$ into an retirement income fund (RIF) which also pays the same interest. You want to be able to withdraw continuously each year from this saving for the next 20 years after retirement. How much will be each withdraw?

c) What would the interest rate r be if you want to continuously withdraw $\$20,000$ per year forever from this $\$200,000$ RIF?

Solution

a) Suppose $FV = 200,000$ after T years.
So, $200000 = \int_0^T 10,000e^{0.08(T-t)}\,dt = \int_0^T 10,000e^{0.08T}e^{-0.08t}\,dt =$
$10,000e^{0.08T}\frac{1}{-0.08}e^{-0.08t}\Big|_0^T = -125,000e^{0.08T}\left(e^{-0.08T}-1\right)$
$= -125,000\left(1-e^{0.08T}\right) \Rightarrow e^{0.08T} = \frac{200,000}{125,000}+1 = 2.6$
$\Rightarrow 0.08T = \ln 2.6 \Rightarrow T = \frac{1}{0.08}\ln 2.6 = 11.94 \approx 12$ years.
Notice that you have earned $\$200,000 - 12\times\$10,000 = \$80,000$ in interest.

b) In this case, the flow rate is constant, and let it be $\$W$ per year, that is $f(t) = W$. Here, $PV = 200,000$. So, $200,000 = \int_0^{20} We^{-0.08t}\,dt =$
$W\frac{1}{-0.08}e^{-0.08t}\Big|_0^{20} = \frac{W}{-0.08}\left(e^{-1.6}-1\right)$
$\Rightarrow W = \frac{-0.08\times 200,000}{e^{-1.6}-1} = 20,047.53$ dollars per year.

c) $200,000 = \int_0^{\infty} 20,000e^{-rt}\,dt = 20,000\frac{1}{-r}\lim_{s\to\infty} e^{-rt}\Big|_0^s$
$= 20,000\frac{1}{-r}\lim_{s\to\infty}\left(e^{-rs}-1\right) = 20,000\frac{1}{-r}(0-1)$
$\Rightarrow r = 0.1$, which is 10%.

Notes:

Example 4.1.5
Suppose that a local car maker is making a new electronic components of a car, and it estimates each component will generate a continuous income stream with flow rate in the t-th year of $10 - 3t$ million dollars per year. Assuming that the component can only last for 10 years, and the money can be invested at the annual rate of 5%, compounded continuously, find the fair market price of each of the components.

Solution In this case the fair market price is the present value of the continuous income stream with

$$f(t) = 10 - 3t, \;\; r = 0.05, \;\; T = 5.$$

So, we have

$$PV = \int_0^T f(t)e^{-rt}\,dt = \int_0^5 (10 - 3t)e^{-0.05t}\,dt = \cdots.$$

Note: This can be integrated using integration by parts.

Notes:

Exercises 4.1

Problems

1. Consider an income stream given by $f(t) = 200$ dollars per year for time $t \geq 5$ years (i.e., beginning after 5 years and lasting forever). Find the present value of this income stream, PV, in term of the interest r, compounded continuously. At what value of r would the present value of this income stream be equal to $1000 \cdot \left[\frac{1-e^{-5r}}{r}\right]$?

2. Assuming that the interest is compounded continuously is r, find the present value of the increasing annuity (a type of continuous income stream in finance) over T years, $f(t) = at$, $0 \leq t \leq T$, where a is a constant.

3. If you invest $f(t) = 2000 + 400t$ dollars per year in a retirement account earning nine percent per year, how much will you have after forty years? How much of that is principal and how much is interest?

4. Suppose that the interest rate remains at 10% compounded continuously for the next ten years. You are offered two options: A) a gold mine producing an income flow of $f(t) = 8 - t$ thousand dollars per year for the next 7 years, after which the mine will be shut down, and B) an annuity paying $g(t) = 7$ thousand dollars per year for 5 years. Which asset should you accept?

5. Your business has the option of buying a piece of machinery that will lower the production costs by $\$10,000$ per year for 7 years. During these years, the machinery will require continuous maintenance and repairs which will cost $1000 + 200t$ dollars per year t years from now. In 7 years, the machinery can be sold for scrap for $\$1000$. Assuming that an effective interest rate of 10%, per annum compounded continuously, what is the maximum amount you should be willing to pay now for the new machinery? Hint: This is the net present value of all saving plus scrap, less expenses.

6. A theatre owner has two alternative planes to renovate his theatre. Plan A calls for immediate cash outlay of $\$250,000$, whereas Plan B calls for $\$180,000$. Adopting Plane A would result in a net income stream given by $f(t) = 630,000$ dollars per year, while Plane B would result in net income stream of $g(t) = 580,000$ dollars per year, both for the next three years. If the future income stream is discounted at 10%, which plan should be the theatre owner choose?

4.2 Consumers' and Producers' Surpluses

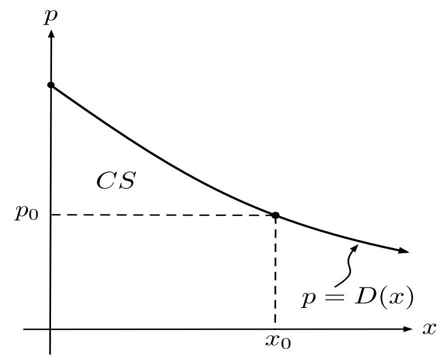


Figure 4.3: The Consumers' Surplus

Let $p = D(x)$ be the **price-demand equation** for a product, where x is the number of items sold at a price p each. Suppose that p_0 is the current price and x_0 is the number of units that can be sold at this price. If the price is higher than p_0, then the demand x is less than x_0. The **consumers' surplus** is defined as the difference between the total amount that consumers are willing and able to pay for a good or service (indicated by the demand curve) and the total amount that they actually do pay (i.e. the market price, p_0).

It can be shown that the consumers' surplus, CS, at the price level of p_0 is defined by

$$CS = \int_0^{x_0} \Big[D(x) - p_0\Big]\, dx \text{ or } CS = \int_0^{x_0} D(x)\, dx - p_0 x_0,$$

which is the area between $p = p_0$ and $p = D(x)$ from $x = 0$ to $x = x_0$. See Figure 4.3

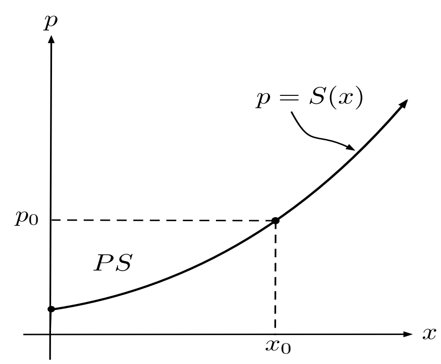


Figure 4.4: The Producers' Surplus

Similarly, if $p = S(x)$ is the **price-supply equation** for a product, p_0 is the current price, and x_0 is the current supply level, then the difference between the amount the producer is willing to supply goods for and the actual amount received by the producer when he/she makes the trade at this price is called producers' surplus **producers' surplus**, PS, which is defined by

$$PS = \int_0^{x_0} \Big[p_0 - S(x)\Big]\, dx \text{ or } PS = p_0 x_0 - \int_0^{x_0} S(x)\, dx,$$

which is the area between $p = p_0$ and $p = S(x)$ from $x = 0$ to $x = x_0$. See Figure 4.3

Notes:

1. In a free market, the price of a product is determined by the relationship between supply and demand.

2. In the case above, the point (x_0, p_0) is the intersection point of $p = D(x)$ and $p = S(x)$. It is called the equilibrium point.

3. The price p_0 is called the equilibrium price and x_0 is called **the equilibrium quantity**

Notes:

4. If a price p^* stabilizes at the equilibrium price p_0, then it is the price level that will determine both the consumers' surplus and the producers' surplus.

5. The total of the consumers' and producers' surplus is called the total gains **the total gains** (from trade). From the graph in the Figure 4.5, we can see that this total gains is the sum of the two areas, between the line $p = p_0$ and $p = D(x)$, and $p = p_0$ and $p = S(x)$. That is the area between the two curves $p = D(x)$ and $p = S(x)$.

   So, the total gains is equal to the sum of the two integrals involved, which is

$$Total\ Gains = CS + PS = \int_0^{x_0} \Big[D(x) - S(x)\Big]\, dx.$$

   It can be easily shown algebraically as follow.

$$CS + PS = \int_0^{x_0} D(x)\, dx - p_0x_0 + p_0x_0 - \int_0^{x_0} S(x)\, dx$$

$$= \int_0^{x_0} D(x)\, dx - \int_0^{x_0} S(x)\, dx = \int_0^{x_0} \Big[D(x) - S(x)\Big]\, dx.$$

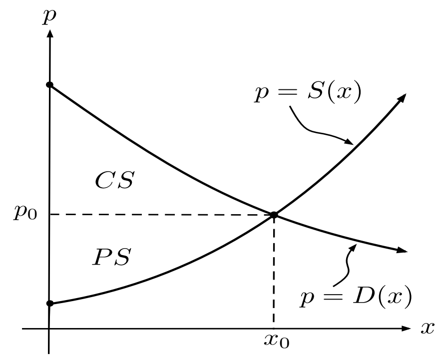


Figure 4.5: The Consumers' and Producers' Surplus

Example 4.2.1

Find the consumers' surplus at a price level of \$8 for the price-demand equation $p = D(x) = 20 - 0.05x$.

Solution Here, $p_0 = 8$. To find x_0, we solve $8 = 20 - 0.05x_0 \Rightarrow x_0 = 240$. Then, $CS = \int_0^{x_0} \Big[D(x) - p_0\Big]\, dx = \int_0^{240} [20 - 0.05x - 8]\, dx$ $= \int_0^{240} (12 - 0.05x)\, dx = \$1,440.$

Example 4.2.2

Find the producers' surplus at a price level of \$20 for the price-demand equation $p = S(x) = 2 + 0.0002x^2$.

Solution Here, $p_0 = 20$. To find x_0, we solve $20 = 2+0.0002x^2 \Rightarrow x_0 = 300$ (we take only the positive x). Then, $PS = \int_0^{x_0} \Big[p_0 - S(x)\Big]\, dx =$ $\int_0^{300} [20 - (2 + 0.0002x^2)]\, dx = \int_0^{300} (18 - 0.0002x^2)\, dx = \$3,600.$

Notes:

Example 4.2.3

Suppose that the price-demand equation and the price-supply equation are given by

$$p = D(x) = 20 - 0.05x, \text{ and } p = S(x) = 2 + 0.0002x^2.$$

Find the consumers' surplus and the producers' surplus at the equilibrium price level.

Solution First we have to find (x_0, p_0). To do this we solve the equation $D(x) = S(x)$, that is $20 - 0.05x = 2 + 0.0002x^2 \Rightarrow 0.0002x^2 + 0.05x - 18 = 0$
$\Rightarrow x^2 + 250x - 90,000 = 0 \Rightarrow x = 200, -450$.
But, the quantity can not be negative. So, the only solution from this equation is $x_0 = 200$. Substituting this into one of the equations, $p = D(x)$ or $p = S(x)$ (do both to double check) gives $p_0 = 10$.
$CS = \int_0^{x_0} \Big[D(x) - p_0\Big]\, dx = \int_0^{200} [20 - 0.05x - 10]\, dx$
$= \int_0^{200} (10 - 0.05x)\, dx = \$1,000$. and
$PS = \int_0^{x_0} \Big[p_0 - S(x)\Big]\, dx = \int_0^{200} [10 - (2 + 0.0002x^2)]\, dx$
$= \int_0^{200} (8 - 0.0002x^2)\, dx = \$1,067$.

Example 4.2.4

Suppose that the price-demand equation and the price-supply equation are given by $p = D(x)$ and $p = S(x)$. The equilibrium price and quantity are $(x_0, p_0) = (200, 100)$. Also, suppose that the amount of the producers' surplus is $\$12,067$, and $\int_0^{200} \Big[D(x) - S(x)\Big]\, dx = 28,067$ (Area between the two curves from $x = 0$ to $x = 200$).

a) What is the amount of the consumers surplus?

b) What does $\int_0^{200} D(x)\, dx$ evaluate to?

Solution

a) What is the amount of the consumers surplus? $\int_0^{200} \Big[D(x) - S(x)\Big]\, dx =$
$\int_0^{200} \Big[D(x) - p_0 + p_0 - S(x)\Big]\, dx = \int_0^{200} \Big[D(x) - p_0\Big]\, dx + \int_0^{200} \Big[p_0 -$

Notes:

$S(x)\Big]\ dx = CS + PS \Rightarrow 28,067 = CS + 12,067 \Rightarrow CS = 16,000$ (dollars) .

b) What does $\int_0^{200} D(x)\ dx$ evaluate to? $CS = \int_0^{200} \Big[D(x) - p_0\Big]\ dx = \int_0^{200} D(x)\ dx - p_0 x_0 = \int_0^{200} D(x)\ dx - (100)(200) \Rightarrow \int_0^{200} D(x)\ dx = CS + (100)(200) = 16,000 + 20,000 = 36,000$ (dollars) .

Notes:

Exercises 4.2

Problems

1. The price-demand and price-supply functions for a good are $p = D(x) = 100 - 0.05x$ and $p = S(x) = 10 + 0.1x$.
   a) Find the equilibrium quantity and price.
   b) What are the consumers' and producers' surplus under this market equilibrium?

2. The price-demand and price-supply functions for a good are $p = D(x) = 215 - x^2$ and $p = S(x) = 10x + 15$.
   a) Find the equilibrium quantity and price.
   b) What are the consumers' and producers' surplus under this market equilibrium?

3. The price-demand and price-supply functions for a good are $p = D(x) = e^{9-x}$ and $p = S(x) = e^{x+3}$.
   a) Find the equilibrium quantity and price.
   b) What are the consumers' and producers' surplus under this market equilibrium?

4. The price-demand and price-supply functions for a good are $p = D(x) = 125 - 0.01x^2$ and $p = S(x) = 0.04x^2$.
   a) Find the equilibrium quantity and price.
   b) What are the consumers' and producers' surplus under this market equilibrium?

5. The price-demand and price-supply functions for a good are $p = D(x) = 20e^{-x}$ and $p = S(x) = 5e^x$.
   a) Find the equilibrium quantity and price.
   b) What are the consumers' and producers' surplus under this market equilibrium?

6. The price-demand and price-supply functions for a good are $p = D(x) = 77 - \dfrac{15x}{\sqrt{x^2+16}}$ and $p = S(x) = 10\ln(4-x) + 50$.
   a) Verify that the equilibrium quantity and price is $(50, 3)$.
   b) What are the consumers' and producers' surplus under this market equilibrium?

4.3 Probabilities

Let's say we want to look at the amount of time spent waiting in line to cross the border into the United States, or the life time of the battery of a random MacBook computer. These quantities are called **continuous random variables** because the have values that will range over an interval of real numbers (even sometimes they are recorded to the nearest integer).

(A) Probability Density Function

A continuous random variable is usually denoted by X. Every continuous random variable, X, has a **probability density function**, $f(x)$, which satisfies the following conditions.

1. $f(x) \geq 0$ for all x.
2. $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx = 1$.

This function can be used to determine the probability that a continuous random variable lies between two values a and b. This **probability** is represented by $P(a \leq X \leq b)$ and is given by

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx.$$

The probability density function can also be used to determine the mean of a continuous random variable. The **mean** (average value) is given by

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx.$$

Note: Another measure of centrality of a probability density function is called **median**. It is a number m such that

$$\int_m^{\infty} f(x)\, dx = \frac{1}{2}.$$

(B) Normal Distributions

In Statistics, most of important random phenomena such as test scores on LPI or heights of individuals from a homogeneous population are modelled by a **normal distribution**. For a normal distribution, a probability density function has the form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)},$$

Notes:

where μ and σ are the mean and **standard deviation** (which measures how spread out the values of the random variable X are). It is a positive constant. Notice that this function has no simple antiderivative. So, integrating it is difficult. But, we make some change to it and use the table.

To do this, we let $u = \dfrac{x-\mu}{\sigma} \Rightarrow du = \dfrac{1}{\sigma}dx$. Thus, we get

$$\int_a^b f(x)\,dx = \int_a^b \frac{1}{\sigma\sqrt{2\pi}}\,e^{-(x-\mu)^2/(2\sigma^2)}\,dx = \int_{u_a}^{u_b} \frac{1}{\sqrt{2\pi}}\,e^{-u^2/2}\,du,$$

where $u_a = \dfrac{a-\mu}{\sigma}, u_b = \dfrac{b-\mu}{\sigma}$. From here, we can make use of the table. Notice that this is a bit different from the probability table in most Statistics courses.

Example 4.3.1

Let $f(x) = \dfrac{x^3}{5000}(10-x)$ for $0 \le x \le 10$ and $f(x) = 0$ for all other values of x.

a) Show that $f(x)$ is a probability density function.

b) Find $P(1 \le X \le 4)$.

c) Find $P(X \ge 6)$.

Solution

a) Since this function is positive or zero on the interval, all we need to do is to show that the integral is one. That is,
$\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{10} \frac{x^3}{5000}(10-x)\,dx = \frac{1}{5000}\int_0^{10}(10x^3 - x^4)\,dx$
$\displaystyle= \frac{1}{5000}\left[\frac{10}{4}x^4 - \frac{1}{5}x^5\right]\Big|_0^{10} = 1.$

b) $\displaystyle P(1 \le X \le 4) = \int_1^4 \frac{x^3}{5000}(10-x)\,dx = \frac{1}{5000}\left[\frac{10}{4}x^4 - \frac{1}{5}x^5\right]\Big|_1^4 =$ 0.08658.

c) $\displaystyle P(X \ge 6) = P(6 \le X \le 10) = \int_6^{10} \frac{x^3}{5000}(10-x)\,dx$
$\displaystyle= \frac{1}{5000}\left[\frac{10}{4}x^4 - \frac{1}{5}x^5\right]\Big|_6^{10} = 0.66304.$

Notes:

Example 4.3.2
Suppose that the probability density function for waiting in line at a Supermarket cashier counter is given by

$$f(t) = \begin{cases} 0, & t < 0 \\ 0.1e^{-\frac{t}{10}}, & t \geq 0 \end{cases}$$

where t is the number of minutes spent waiting in line.

a) Verify that this is a probability density function.

b) Determine the probability that a person will wait in line for at least 6 minutes.

c) Determine the average waiting time and the median waiting time for a wait in the line.

d) Find the median m of waiting time.

Solution

a) Since this function is positive or zero on the interval, all we need to do is to show that the integral is one. That is,
$\int_{-\infty}^{\infty} f(t)\,dt = \int_0^{\infty} 0.1e^{-\frac{t}{10}}\,dt = \lim_{s\to\infty} \int_0^{s} 0.1e^{-\frac{t}{10}}\,dt$
$= \lim_{s\to\infty} \Big[-e^{-\frac{t}{10}}\Big]\Big|_0^s = \lim_{s\to\infty} \Big[-e^{-\frac{s}{10}} + 1\Big] = 1.$

b) $P(T \geq 6) = \int_6^{\infty} 0.1e^{-\frac{t}{10}}\,dt = \lim_{s\to\infty} \Big[-e^{-\frac{t}{10}}\Big]\Big|_6^s$
$= \lim_{s\to\infty} \Big[-e^{-\frac{s}{10}} + e^{-\frac{6}{10}}\Big] \approx 0.548812.$

c) $\int_{-\infty}^{\infty} tf(t)\,dt = \int_0^{\infty} 0.1te^{-\frac{t}{10}}\,dt = \lim_{s\to\infty} \int_0^{s} 0.1te^{-\frac{t}{10}}\,dt$
$= \lim_{s\to\infty} \Big[-(t+10)e^{-\frac{t}{10}}\Big]\Big|_0^s = \lim_{s\to\infty} \Big[-(s+10)e^{-\frac{s}{10}} + 10\Big] \Longrightarrow \mu =$ 10, minutes. Note that we use integral by parts here.

d) $\int_m^{\infty} f(t)\,dt = \frac{1}{2} \Rightarrow \lim_{s\to\infty} \Big[-e^{-\frac{t}{10}}\Big]\Big|_m^s = \lim_{s\to\infty} \Big[-e^{-\frac{s}{10}} + e^{-\frac{m}{10}}\Big]$
$= \frac{1}{2} \Rightarrow e^{-\frac{m}{10}} = \frac{1}{2} \Rightarrow -\frac{m}{10} = \ln\left(\frac{1}{2}\right) \Rightarrow m = (-10)\cdot(\ln 2).$

Notes:

Example 4.3.3

Let $f(x) = kx\sqrt{16 - x^2}$ for $0 \le x \le 4$ and $f(x) = 0$ for all other values of x.

a) Find the value of k so that $f(x)$ is a probability density function.

b) For this value of k, find $P(X < 2)$.

Solution

a) $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_0^4 f(x)\,dx = 1 \Rightarrow k = \frac{3}{64}$.

b) $\displaystyle P(X < 2) = \int_0^2 \frac{3}{64} x\sqrt{16 - x^2}\,dx = -\frac{3}{128}\left(\frac{2}{3}(16 - x^2)^{3/2}\right)\Big|_0^2 \approx$ 0.3509.

Example 4.3.4

A survey shows that the heights of adult males in the United States are normally distributed with mean 69.0 inches and standard deviation 2.8 inches.

a) What is the probability that an adult male chosen at random is between 65 inches and 73 inches tall? Write your answer in integral form.

b) What percentage of the adult male population is more than 6 feet (or 72 inches) tall? Write your answer in integral form.

Solution

a) $\displaystyle P(65 \le X \le 73) = \int_{65}^{73} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)}\,dx = \int_{-1.43}^{1.43} \frac{1}{\sqrt{2\pi}}\, e^{-u^2/2}\,du =$ $\displaystyle 2\int_0^{1.43} \frac{1}{\sqrt{2\pi}}\, e^{-u^2/2}\,du \Rightarrow P(65 \le X \le 73) = 2 \cdot (0.4236) = 0.8472.$

b) $\displaystyle P(X > 72) = 1 - P(0 \le X \le 72) = 1 - \int_0^{72} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)}\,dx =$ $\displaystyle 1 - \int_{-24.64}^{1.07} \frac{1}{\sqrt{2\pi}}\, e^{-u^2/2}\,du = 1 - \left(\int_{-24.64}^{0} \frac{1}{\sqrt{2\pi}}\, e^{-u^2/2}\,du + \int_0^{1.07} \frac{1}{\sqrt{2\pi}}\, e^{-u^2/2}\,du\right)$ $= 1 - (0.5 + 0.3577) = 1 - 0.8577 = 0.1423.$

Notes:

Example 4.3.5
For any normal distribution, find the probability that the random variable lies within two standard deviations of the mean. Write your answer in integral form.

Solution $P(\mu-2\sigma \leq X \leq \mu+2\sigma) = \int_{\mu-2\sigma}^{\mu+2\sigma} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)}\, dx =$
$\int_{-2}^{2} \frac{1}{\sqrt{2\pi}}\, e^{-u^2/2}\, du = 2\int_{0}^{2} \frac{1}{\sqrt{2\pi}}\, e^{-u^2/2}\, du \approx 0.9545.$

Note: For normal distribution, we use the table handed out in class to read off the values of the probabilities in example 4 and 5 above.

Notes:

Exercises 4.3

Problems

1. Consider an income stream given by $f(t) = 200$ dollars per year for time $t \geq 5$ years (i.e., beginning after 5 years and lasting forever). Find the present value of this income stream, PV, in term of the interest r, compounded continuously. At what value of r would the present value of this income stream be equal to $1000 \cdot \left[\frac{1-e^{-5r}}{r}\right]$?

2. Assuming that the interest is compounded continuously is r, find the present value of the increasing annuity (a type of continuous income stream in finance) over T years, $f(t) = at$, $0 \leq t \leq T$, where a is a constant.

3. If you invest $f(t) = 2000 + 400t$ dollars per year in a retirement account earning nine percent per year, how much will you have after forty years? How much of that is principal and how much is interest?

4. Suppose that the interest rate remains at 10% compounded continuously for the next ten years. You are offered two options: A) a gold mine producing an income flow of $f(t) = 8 - t$ thousand dollars per year for the next 7 years, after which the mine will be shut down, and B) an annuity paying $g(t) = 7$ thousand dollars per year for 5 years. Which asset should you accept?

5. Your business has the option of buying a piece of machinery that will lower the production costs by $\$10,000$ per year for 7 years. During these years, the machinery will require continuous maintenance and repairs which will cost $1000 + 200t$ dollars per year t years from now. In 7 years, the machinery can be sold for scrap for $\$1000$. Assuming that an effective interest rate of 10%, per annum compounded continuously, what is the maximum amount you should be willing to pay now for the new machinery? Hint: This is the net present value of all saving plus scrap, less expenses.

6. A theatre owner has two alternative planes to renovate his theatre. Plan A calls for immediate cash outlay of $\$250,000$, whereas Plan B calls for $\$180,000$. Adopting Plane A would result in a net income stream given by $f(t) = 630,000$ dollars per year, while Plane B would result in net income stream of $g(t) = 580,000$ dollars per year, both for the next three years. If the future income stream is discounted at 10%, which plan should be the theatre owner choose?

5: Multivariate Calculus

5.1 The Three-Dimensional Coordinate System

Recall that the Cartesian plane is determined by two perpendicular axes, that is, by the x-axis and the y-axis. They meet at the origin. This gives the two-dimensional coordinate system. From here we can construct a three-dimensional coordinate system by constructing a third axis called the z-axis perpendicular to the Cartesian plane at the origin. There are two ways of orienting the coordinate system. The one we use here is based on the right-handed system.

If you are standing at the origin (or at a corner of a room), the z-axis is along the direction of your head, your right hand is pointing along the x-axis and the left hand is pointing along the y-axis. See the Figure 5.1 (a). As you can see from Figure 5.1 (b), there are three coordinate planes: the xy-plane, the xz-plane, and the yz-plane.

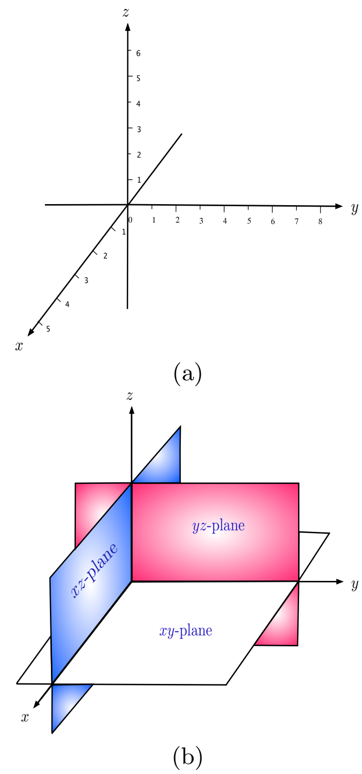


Figure 5.1: 3D Coordinate System

Points In Space: A point P in space, in this three-dimensional system, is determined by a ordered triple (x, y, z), where x, y, and z are as follows.

- $x =$ directed distance from the yz-plane to the point P.
- $y =$ directed distance from the xz-plane to the point P.
- $z =$ directed distance from the xy-plane to the point P.

Example 5.1.1

a) Plot the point $P(4, 5, 6)$.

b) Find the coordinates of the vertices A, B, and C of the following rectangular box.

Solution

a) See the Figure 5.2 (a).

b) $A(0, -10, 4)$, $B(-3, -10, 4)$, and $C(-3, -10, 0)$. See the Figure 5.2 (b)

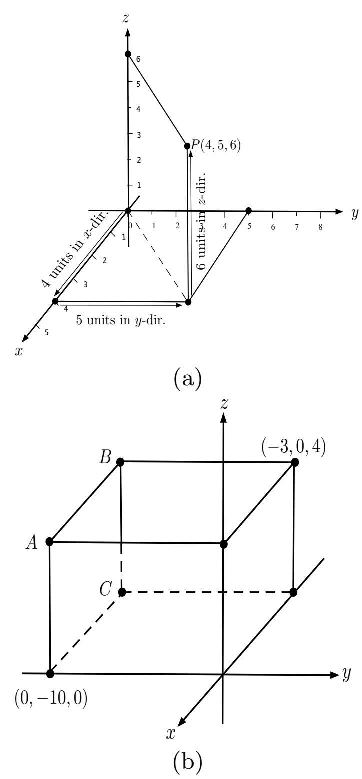


Figure 5.2: Example 5.1.1.

Example 5.1.2

For each of point $P(x, y, z)$ given below, let $A(x, y, 0)$, $B(x, 0, z)$, and $C(0, y, z)$ be points in the xy-, xz-, and yz-planes. Plot and label the points P, A, B, and C.

a) $P(-3, 2, 4)$

b) $P(-2, -4, -3)$

Solution

a) $P(-3, 2, 4) \Rightarrow A(-3, 2, 0),\ B(-3, 0, 4),\ C(0, 2, 4)$. See Figure 5.3 (a).

b) $P(-2, -4, -3) \Rightarrow A(-2, -4, 0),\ B(-2, 0-3),\ C(0, -4, -3)$. See Figure 5.3 (b).

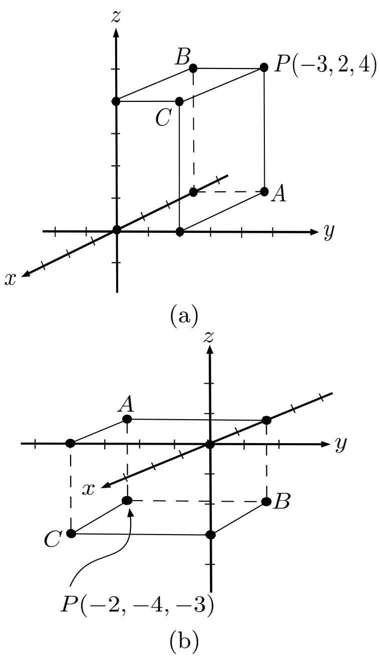


Figure 5.3: Example 5.1.2.

Midpoint and Distance Formula in 3D: Suppose that the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points in space. Let M is the midpoint of the line segment PQ. Its coordinates are given by

$$\left(x_M, y_M, z_M\right) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

See Figure 5.4. For instance, the midpoint of the line segment joining the origin and the point $P(4, 5, 6)$ is $\left(x_M, y_M, z_M\right) = \left(\frac{0+4}{2}, \frac{0+5}{2}, \frac{0+6}{2}\right) = \left(2, \frac{5}{2}, 3\right)$. See Figure 5.4.

Distance Formula: Recall, in xy-plane, the distance between point $P(x_1, y_1)$ and $Q(x_2, y_2)$, denoted by $|PQ|$, is

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

In the xyz-space is similar. We just add the z component to the above formula. Thus, the distance between the point $P(x_1, y_1, z_1)$ and the point $Q(x_2, y_2, z_2)$, denoted by $|PQ|$, is

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

For instance, the distance from the origin to the point P in the previous figure is

$$|OP| = \sqrt{(4-0)^2 + (5-0)^2 + (6-0)^2} = \sqrt{16 + 25 + 36} = \sqrt{77}.$$

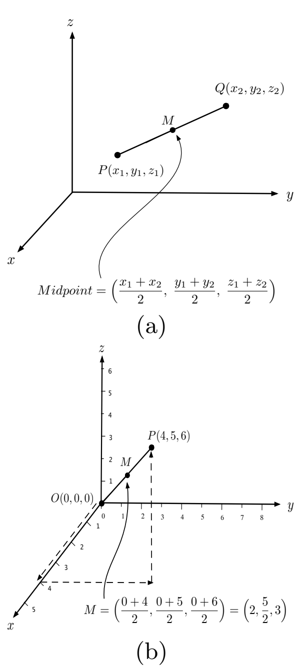


Figure 5.4: Midpoint in 3D.

Notes:

Spheres and Balls: Sphere is the set of all points in space that are a fixed distance r, which is called radius, away from a fixed point (a, b, c), known as centre. The set of all points inside and on this sphere forms a ball. The equation of the sphere and ball above are given by

$$\text{Sphere} : (x-a)^2 + (y-b)^2 + (z-c)^2 = r^2,$$

$$\text{Ball} : (x-a)^2 + (y-b)^2 + (z-c)^2 \leq r^2.$$

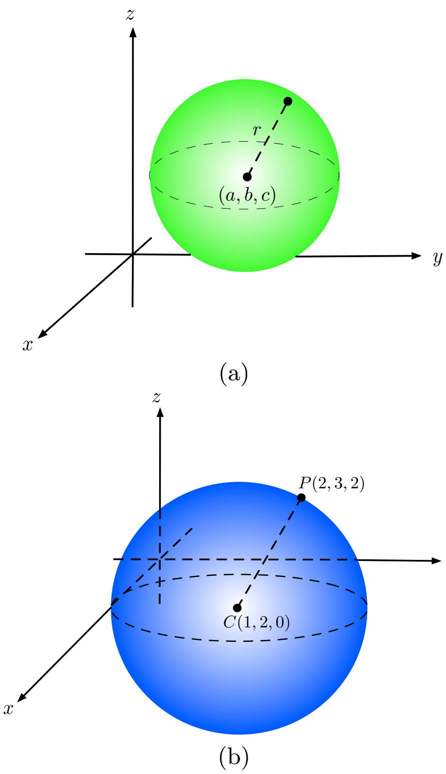


Figure 5.5: Spheres and Balls.

Example 5.1.3

a) Find an equation of a sphere with centre $C(1, 2, 0)$ passing through the point $P(2, 3, 2)$.

b) Find inequality that describes the ball centre $C(1, 2, -2)$ and radius 2.

c) Give a geometric description of the $x^2+y^2+z^2-6x+6y-8z-2 = 0$.

d) Find an equation of a sphere passing through $P(-4, 2, 3)$ and $Q(0, 2, 7)$ with its centre at the midpoint of PQ.

Solution

a) $(x-1)^2+(y-2)^2+(z-0)^2 = r^2$, where $r = \sqrt{(2-1)^2 + (3-2)^2 + (2-0)^2} = \sqrt{6}$. So, we have

$$(x-1)^2 + (y-2)^2 + (z-0)^2 = 6.$$

b) $(x-1)^2 + (y-2)^2 + (z+2)^2 \leq 4$.

c) Completing squares in each variable, gives

$$(x^2-6x)+(y^2+6y)+(z^2-8z) = 2 \Rightarrow (x-3)^2+(y+3)^2+(z-4)^2 = 36 = 6^2.$$

It is a sphere centre at $(3, -3, 4)$ with radius $r = 6$.

d) The centre is at the midpoint $M(-2, 2, 5)$. The equation is $(x+2)^2 + (y-2)^2 + (z-5)^2 = r^2$, where r is given by $r = \frac{1}{2}\sqrt{16+0+16} = \frac{1}{2}\sqrt{32} = \sqrt{2}$. So, we have

$$(x+2)^2 + (y-2)^2 + (z-5)^2 = 2.$$

Notes:

Exercises 5.1

Problems

1. In a three-dimensional coordinate system, starting at the origin, say you move backward 3 units along the x-axis, forward 4 units on the y-axis and upward one unit: what is the location of your position (coordinates)? How far have you travelled? How far are you from the origin?

2. For the point $(3, 2, 1)$:

   a) What is its projection of the point $(3, 2, 1)$ onto the xy-, xz- and yz-planes?

   b) In a 3-dimensional coordinate system, draw a rectangular box showing the point $(3, 2, 1)$ at one vertex, the origin opposite, and label all the other six points, identifying the projections from (a). Find the length of the box's diagonal.

3. Consider the points $P_1(1, 2, 1)$ and $P_2(7, -3, 4)$.

   a) Find the midpoint $P_{1,2}$ between P_1 and P_2.

   b) Show that every point P on the line through P_1 and P_2 satisfies $(1 + 6t, 2 - 5t, 1 + 3t)$, where t is a real-valued **parameter**, by checking that $P(t = 0) = P_1$ and $P(t = 1) = P_2$.

   c) Does the point $Q(2, 7, -5)$ lie on the same line as P_1 and P_2? Explain.

In Exercises 4 – 7, show that the equation represents a sphere and give its centre and radius.

4. $x^2 + y^2 + z^2 = 4x + 2y - 6z + 2$.

5. $x^2 + y^2 + z^2 + x + y + z = 0$.

6. $2x^2 + 2y^2 + 2z^2 + 16 = 4x + 12y$

In Exercises 7 – 11, find the equation of the circles formed by the intersection of the sphere $x^2 + y^2 + z^2 = 100$ **with:**

7. the plane $x = 8$.

8. the plane $y = 4$.

9. the plane $z = 10$.

.

5.2 Planes and Surfaces

Functions of one independent variable such as $f(x) = \sin x$ or equations in two variables x and y such as $x^2 + y^2 = 1$ describe curves in 2-D. In 3-D, functions with two independent variables such as $f(x, y) = x^2 - 3y^2$ or equations in two variables $x^2 + y^2 - 3z^2 = 4$ describe **surfaces**.

Equations of Planes in Space: The general equation of a plane in space is

$$ax + by + cz = d,$$

where a, b, c and d are constants. As we can see from the figure below the plane intersects the three coordinate axes at the three intercepts. that is, x-, y-, and z-intercepts. The triangular shape whose vertices are those intercepts is part of the plane. it helps us to visualize the plane. Also, it intersects each of the xy-, xz-, and yz-planes in three lines. These lines are called traces of the plane. See Figure 5.6.

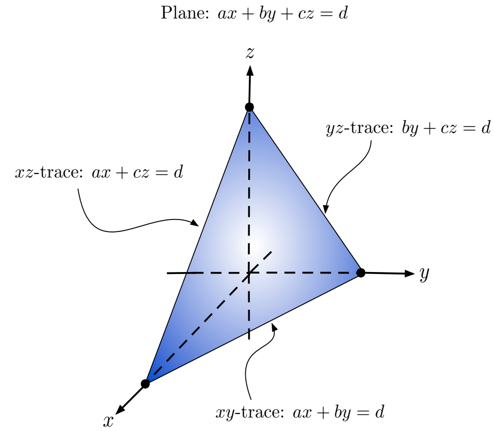


Figure 5.6: Equation of Planes in Space.

In point-slope form, a non-vertical plane passing through a point $P(x_1, y_1, z_1)$ with slope m in the x-direction and slope n in the y-direction is given by

$$z - z_1 = m(x - x_1) + n(y - y_1).$$

Example 5.2.1
Find an equation of a plane passing through three points $P(1, 1, 1)$, $Q(2, 0, 3)$, and $R(1, 5, 9)$.

Solution Point-Slope form gives $z - 1 = m(x - 1) + n(y - 1)$. It passes through Q implies that $3 - 1 = m(2 - 1) + n(0 - 1) \Rightarrow m - n = 2$. Also, it passes through R implies that $9 - 1 = m(1 - 1) + n(5 - 1) \Rightarrow 4n = 8 \Rightarrow n = 2$.Since $m - n = 2$, we get $m = 4$. So, the equation of the plane is $z - 1 = 4(x - 1) + 2(y - 1) \Rightarrow z = 4x + 2y - 5$ or in general form is $4x + 2y - z = 5$.

Example 5.2.2
Sketch each of the following planes.

a) The plane that passes through $(3, 0, 0)$, $(0, 6, 0)$ and $(0, 0, 5)$.

b) The $x = y$ plane.

Solution See the following graphs in Figure 5.7.

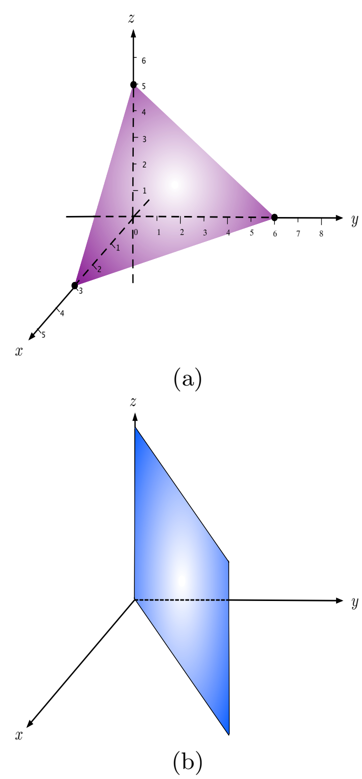


Figure 5.7: Example 5.2.2.

Notes:

Notice that the equation $y = x$ above describes all set of points in space that have the same x and y coordinates for every z coordinate.

Example 5.2.3

Sketch each of the following planes.

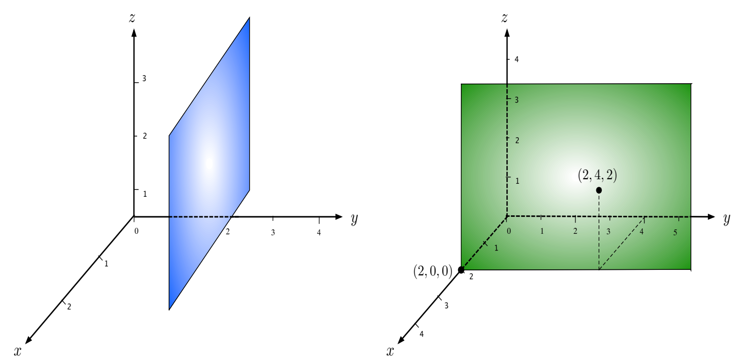
a) The plane parallel to the xz-plane containing $(1, 2, 3)$, and find its equation.

b) The plane parallel to the yz-plane through the point $(2, 4, 2)$ and find its equation.

Solution

a) The equation is the plane $y = 2$.

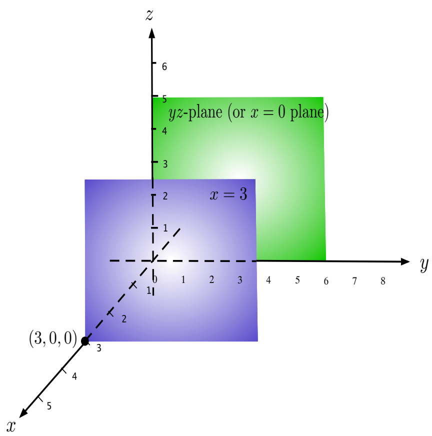
b) The equation is the plane $x = 2$.



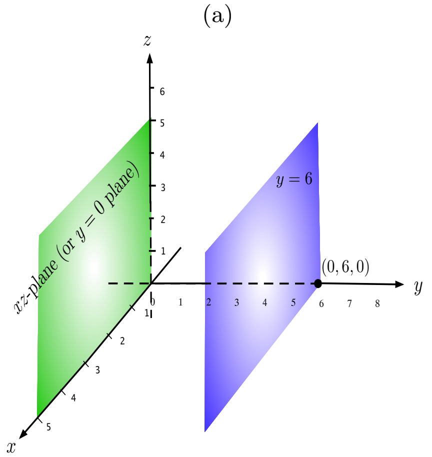
(a) (b)

Figure 5.8: Example 5.2.3.

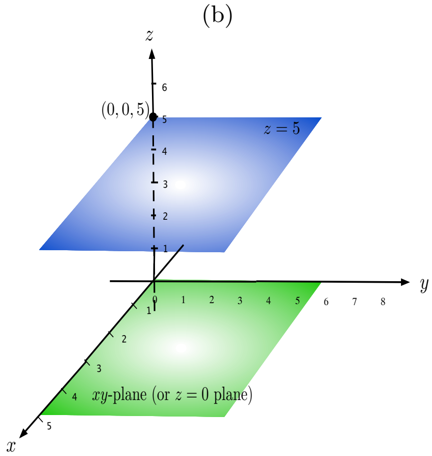
For more examples, see Figure 5.9.



(a)



(b)



(b)

Figure 5.9: The $x = 3$, $y = 6$, and $z = 5$ planes.

Notes:

Cylinders and Traces:

(1) Cylinder: Given a curve C in a plane, and a line L not in the plane, a cylinder is the surface consisting of all lines parallel to the line L that passes through the curve C. See Figure 5.10 (a).

Notice that when the line L is parallel to one of the coordinate axes, the cylinder is also parallel to one of the coordinate axes. For instance, in 3-D the equation $y = x^2$ describes the cylinder consisting of all lines parallel to the z-axis that pass through the parabola $y = x^2$ in the xy-plane. Likewise, the equation $y = z^2$ describes a cylinder parallel to the x-axis. See Figure 5.10. (b) and (c).

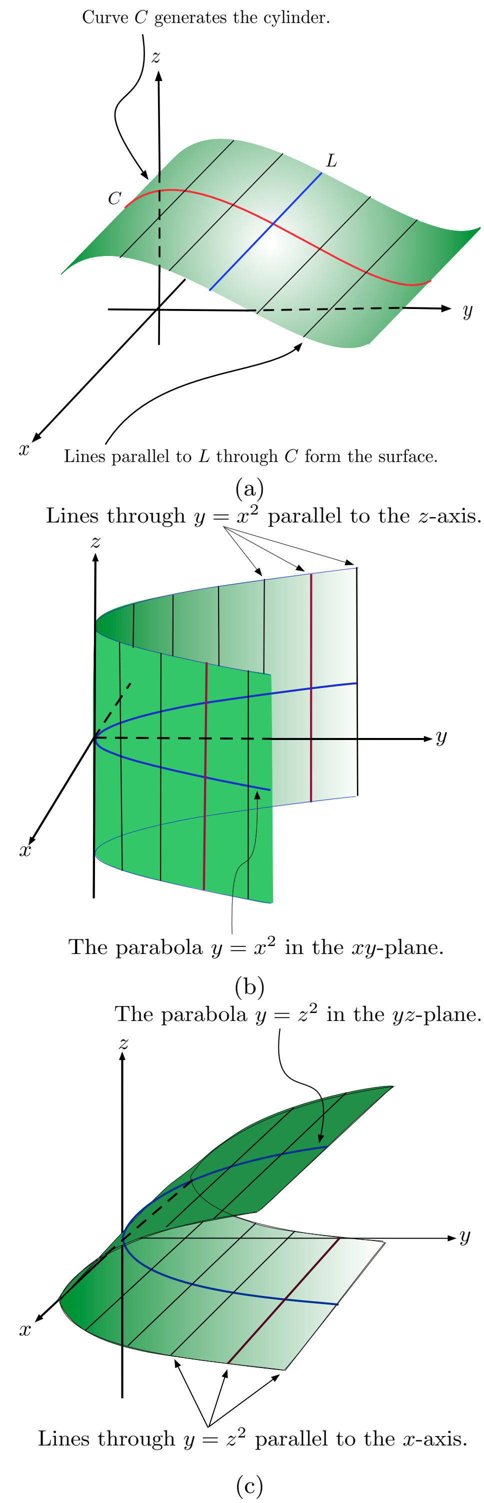


Figure 5.10: Cylinders.

Notes:

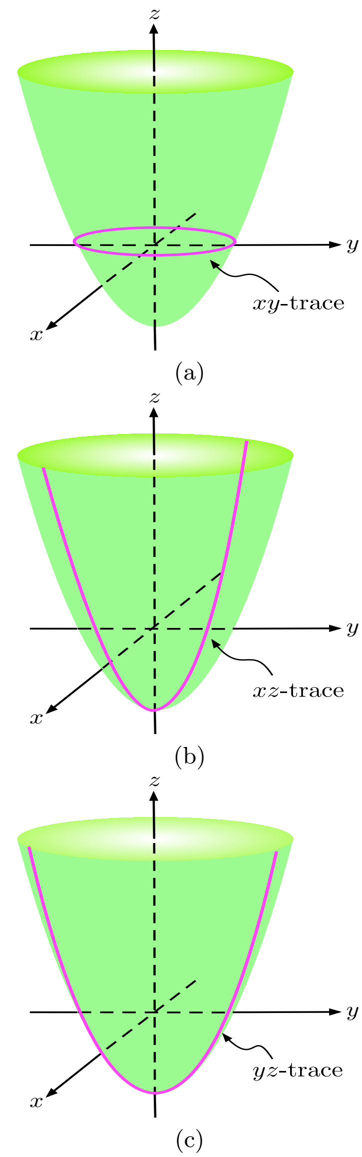


Figure 5.11: Traces.

(2) Traces: A trace of a surface is the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes. The traces in the coordinate planes are called the xy-, the xz-, and the yz-traces. See Figure 5.11.

Notes:

Quadric Surfaces:

So far, we have seen three types of surfaces. They are spheres, planes, and cylinders. Another common type of surfaces in space is a quadric surface, whose equation is of the form

$$Ax^2 + By^2 + Cz^2 + Dx + Ey + Fz + G = 0,$$

where the coefficients A, B, C, D, E, F, and G are constants and not all of them are zero.

There are six basic types of quadric surfaces.

1. Elliptic Cone: Vertex at $(0, 0, 0)$. See Figure 5.12 (a)
2. Elliptic Paraboloid: Vertex at $(0, 0, 0)$. See Figure 5.12 (b)
3. Ellipsoid: Centre at $(0, 0, 0)$. See Figure 5.12 (c)
4. Hyperbolic Paraboloid: Vertices at $(0, 0, 0)$. See Figure 5.14 (a)
5. Hyperboloid of One Sheet: Vertices at $(\pm a, 0, 0)$ and $(0, \pm b, 0)$. See Figure 5.14 (b)
6. Hyperboloid of Two Sheet: Vertices at $(0, 0, \pm c)$. See Figure 5.14 (c)

Notice that all of the basic quadric surfaces above are centred at the origin or have vertices at the origin and have axes along the coordinate axes. Also, only one type of orientation of the surfaces is shown. If a quadric surface has a different centre or is oriented along a different axis, then the standard equation shown must be modified accordingly using transformations and complete squares (in three variables) that we have learned in any precalculus course.

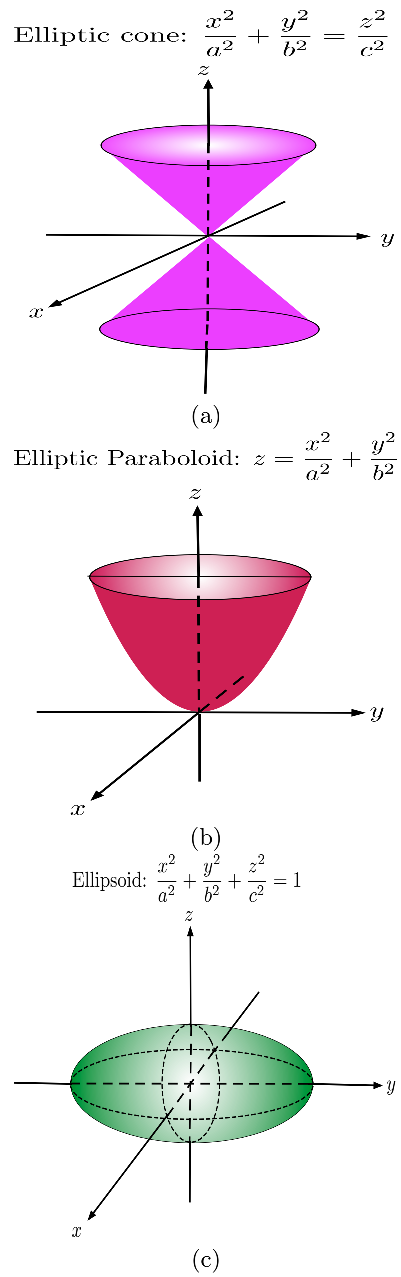


Figure 5.12

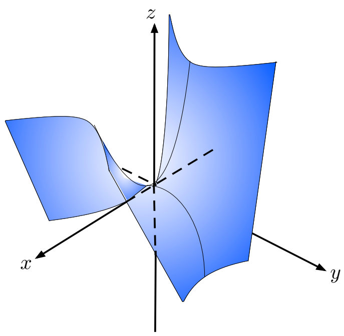
Example 5.2.4

The elliptic cone $\frac{x^2}{2^2} + \frac{y^2}{3^2} = \frac{z^2}{5^2}$ has centre at $(0, 0, 0)$. What is the centre of the elliptic cone $\frac{(x-1)^2}{2^2} + \frac{(y+3)^2}{3^2} = \frac{(z-2)^2}{5^2}$?

Solution The elliptic cone $\frac{(x-1)^2}{2^2} + \frac{(y+3)^2}{3^2} = \frac{(z-2)^2}{5^2}$ has centre at $(1, -3, 2)$.

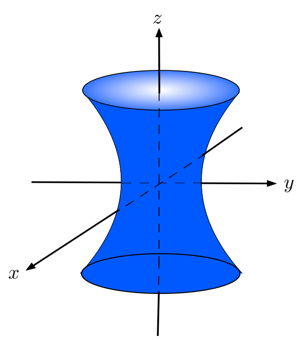
Notes:

Hyperbolic paraboloid: $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$



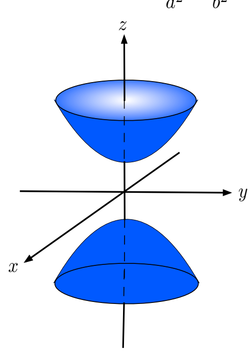
(a)

Hyperboloid of one sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$



(b)

Hyperboloid of two sheets: $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



(c)

Figure 5.14

Example 5.2.5

Consider the equation $-4x^2 - 16y^2 + z^2 + 64y - 80 = 0$. What quadric surface does it represent?

Solution Completing square on the y variable gives

$$-4x^2 - 16(y^2 - 4y) + z^2 - 80 = 0 \Rightarrow -4x^2 - 16(y-2)^2 + z^2 = 16$$

$$\Rightarrow -\frac{x^2}{4} - \frac{(y-2)^2}{1} + \frac{z^2}{16} = 1.$$

This is a hyperboloid of two sheets with vertices at $(0, 2, \pm 4)$.

Example 5.2.6

Suppose we have $x^2 - \frac{y^2}{3} + 9z^2 - 1 = 0$. Find all the intercepts, if any. Find the equations of the xy-, the xz-, and the yz-traces, if any. Then, sketch a graph of this surface.

Solution For x-int., set $y = z = 0$. This gives $x^2 = 1 \Rightarrow x = \pm 1$. Likewise, we can see that the z-int., is $z = \pm\frac{1}{3}$. But, there is no y-int.

For the equations of the traces, we set $z = 0$, $y = 0$, and $x = 0$ respectively in the equation to get

$$x^2 - \frac{y^2}{3} - 1 = 0,\ x^2 + 9z^2 - 1 = 0,\ -\frac{y^2}{3} + 9z^2 - 1 = 0.$$

This surface is the hyperboloid of one sheet along the y-axis. See Figure 5.13.

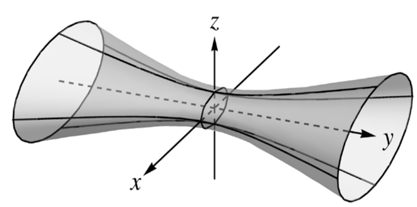


Figure 5.13: Quadric Surface in Example 5.2.6.

Notes:

Exercises 5.2

Problems

1. Answer TRUE or FALSE and give a brief explanation (this could be a quick sketch):
   a) Two planes that are parallel to a third plane are parallel.
   b) Two planes that are perpendicular to a third plane are parallel.
   c) Two planes that are parallel to a line are parallel.
   d) Two planes that are perpendicular to a line are parallel.

In Exercises 2 – 4, find the equation of the plane satisfying the given conditions:

2. The plane that contains the points: $(0, 1, 1)$, $(1, 0, 1)$ and $(1, 1, 0)$.

3. The plane that contains the points: $(6, 3, 2)$, $(5, 3, 7)$ and $(1, 1, 1)$.

4. The plane through the origin parallel to the plane $x - 3y + 6z = 4$.

5. Find two points on the line of intersection of the planes $x + y + z = 6$ and $x - y + z = 0$.

6. Find the distance between the parallel planes $2x + 4y - 2z = 7$ and $-x - 2y + z = 12$. HINT: the distance between planes $ax + by + cz = d_1$ and $ax + by + cz = d_1$ is: $D = |d_1 - d_2|/(a^2 + b^2 + c^2)^{1/2}$.

7. Let $z = f(x, y) = \ln(1 + x + y)$.
   a) Find the domain of f and sketch it on an xy-coordinate system, shading appropriately.
   b) Find the range of f.

8. Let $z = g(x, y) = \ln(1 + x)$.
   a) Sketch $z = \ln(1 + x)$ on an xz-coordinate system.
   b) g is a cylinder: explain how the graph in part (a) can be used to generate its surface.
   c) Find the domain of g and sketch it on an xy-coordinate system, shading appropriately.

9. The surface $z = h(x, y) = \ln(1 + x + y)$ is also a cylinder, but it is not as obvious to see that.
   a) Let $s = x + y$. What is the value of h when $s = 0$? When $s = 1$? When (if) $s = -1$? Explain why keeping $x + y$ constant will also keep $z = \ln(1 + x + y)$ at the same z-value. Sketch the graph of $s = x + y$ for several different s values.
   b) Sketch the graph of $z = \ln(1 + s)$ on an sz-coordinate system.
   c) Explain how the graph in part (b) can be used to generate the surface of h.

.

5.3 Functions of Several Variables

Functions of Two Variables:

> **Definition 5.3.1 Function of Two Variables**
>
> A function, f, of two variables is a rules that assigns to each ordered pair of real numbers (x, y) in a set D called domain a unique real number z in a set called range (a subset of real number $\mathbb{R}$).

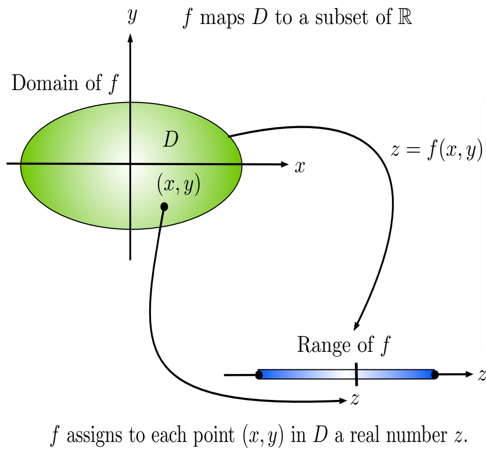


Figure 5.15: Function of Two Variables : $z = f(x, y)$.

Notice that the variables x and y are independent variables, and z is dependent variable. Domain is of a function $f(x, y)$ is the set of all ordered pairs at which it is defined.

Example 5.3.1

Let $f(x, y) = x^2 - y^2 + 5$. Find the followings.

a) $f(-2, 4)$ and $f(-1, -2)$.

b) $\dfrac{f(x+h, y) - f(x, y)}{h}$.

c) $\dfrac{f(x, y+h) - f(x, y)}{h}$.

Solution

a) $f(-2, 4) = (-2)^2 - 4^2 + 5 = 4 - 16 + 5 = 7$ and
$f(-1, -2) = (-1)^2 - (-2)^2 + 5 = 1 - 4 + 5 = 2$.

b) $\dfrac{f(x+h, y) - f(x, y)}{h} = \dfrac{[(x+h)^2 - y^2 + 5] - [x^2 - y^2 + 5]}{h}$
$= \dfrac{x^2 + 2xh + h^2 - y^2 + 5 - x^2 + y^2 - 5}{h} = \dfrac{2xh + h^2}{h} \Rightarrow \dfrac{f(x+h, y) - f(x, y)}{h} =$
$2x + h$.

c) $\dfrac{f(x, y+h) - f(x, y)}{h} = \dfrac{[x^2 - (y+h)^2 + 5] - [x^2 - y^2 + 5]}{h}$
$= \dfrac{x^2 - y^2 - 2yh - h^2 + 5 - x^2 + y^2 - 5}{h} = \dfrac{-2yh - h^2}{h} \Rightarrow \dfrac{f(x, y+h) - f(x, y)}{h} =$
$-2y - h$.

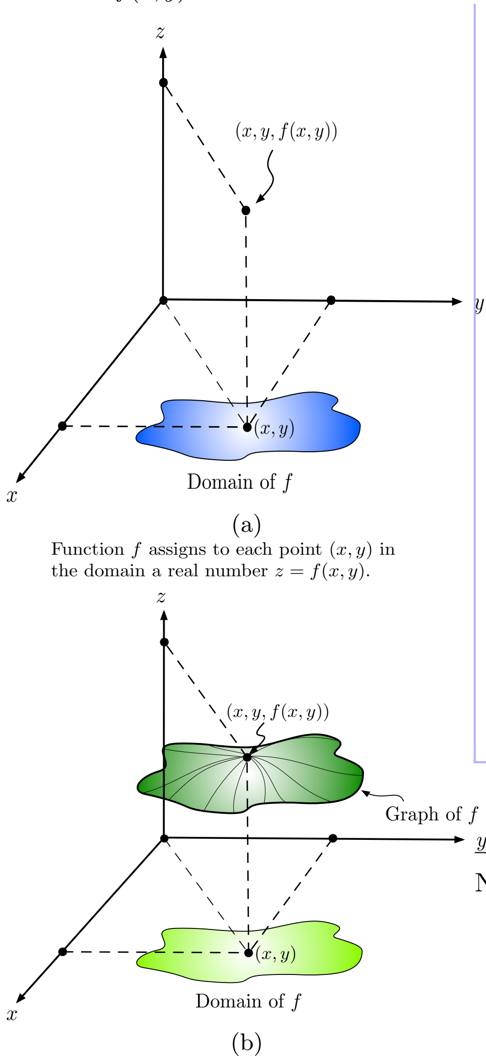


Figure 5.16: Graph Function of Two Variables.

Notes:

Definition 5.3.2 Graph of Function of Two Variables

The graph of a function f of two variables x and y is the set of points (x, y, z) that satisfy the equation $z = f(x, y)$. That is, for each point (x, y) in the domain of f, the point $(x, y, f(x, y))$ lies on the graph of f.

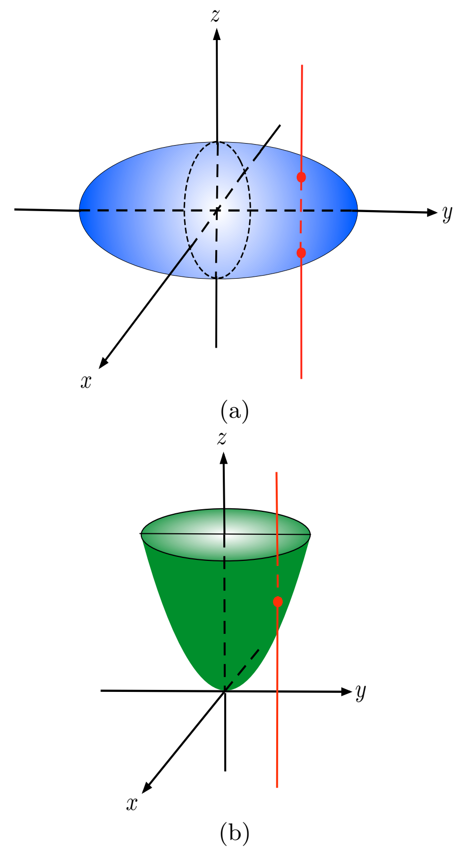


Figure 5.17: Vertical line test for function.

Just like functions of one variable, functions of two variables must pass a vertical line test. The Figure 5.17 (a) is not a graph of a function, but the Figure 5.17 (b) is.

Example 5.3.2
Find the domain of each of the following functions.

a) $f(x, y) = \sqrt{9 - x^2 - y^2}$.

b) $g(x, y) = \ln(x - y^2)$.

Solution

a) $9 - x^2 - y^2 \geq 0 \Rightarrow x^2 + y^2 \leq 9$. So, the domain is $\{(x, y)/x^2 + y^2 \leq 9\}$, which is the set of all points on or inside the circle centred at the origin in the xy-plane with radius 3. See the Figure 5.18.

b) $x - y^2 > 0 \Rightarrow x > y^2$. So, the domain is $\{(x, y)/x > y^2\}$.

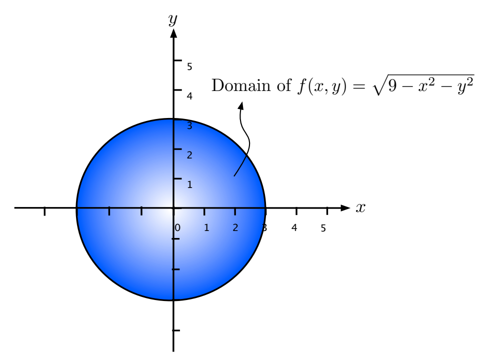


Figure 5.18: Example 5.3.2

Notes:

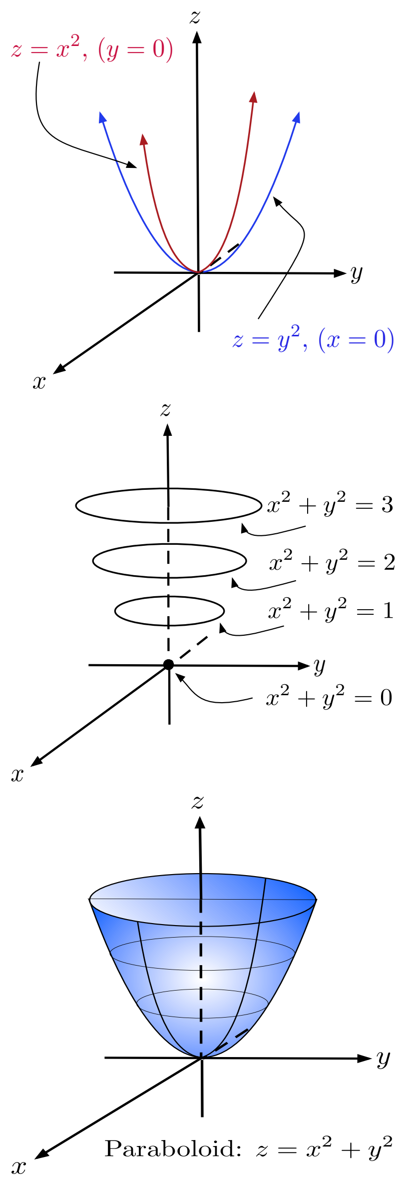


Figure 5.20: Example 5.3.3

Example 5.3.3

Sketch the graph of the function $z = f(x, y) = x^2 + y^2$.

Solution Trace in the yz-plane, $x = 0$ is $z = y^2$, a parabola. Trace in the xz-plane, $y = 0$ is $z = x^2$, a parabola. Trace in the xy-plane, $z = 0$ is $x^2 + y^2 = 0$, which is a single point $(0, 0)$. But, if $z = k > 0$, $x^2 + y^2 = k$, which is a circle centre at $(0, 0)$ with radius $\sqrt{k}$ in the xy-plane, but lifted to the height $z = k$. See Figure 5.20. It is a paraboloid, which is the same as elliptic paraboloid when $a = b = 1$.

Example 5.3.4

How do you sketch the graph of the function $z = x^2 + y^2 - 2x - 4y + 8$.

Solution Completing squares on the x and y gives $z = (x - 1)^2 + (y - 2)^2 + 3$. The graph of this is obtained by shifting the graph of $z = x^2 + y^2$ 1 unit in the x-direction, 2 units in the y-direction, and 3 units up. The new vertex is at $(1, 2, 3)$.

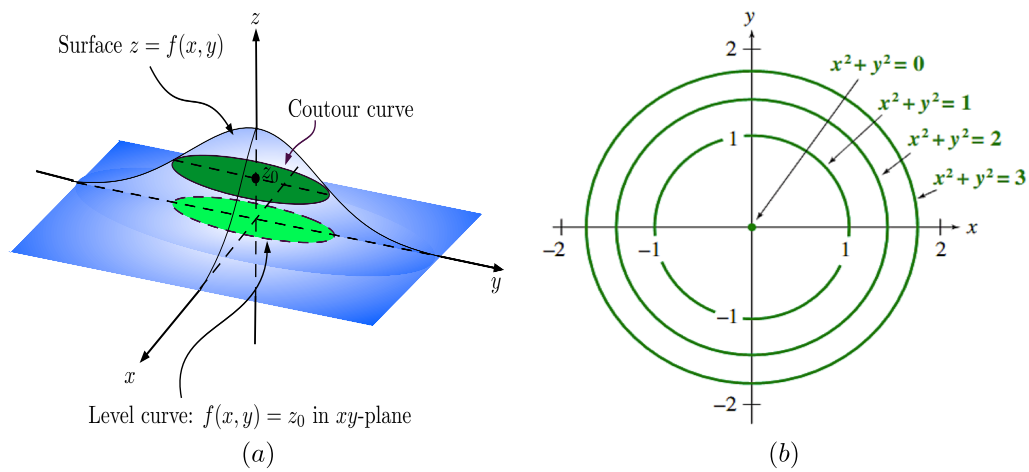


Figure 5.19: Level Curves.

Level Curves:

In example 5.3.3, we can see that the circles $x^2 + y^2 = 1$, $x^2 + y^2 = 2$, $x^2 + y^2 = 3$ are obtained algebraically by letting $z = 1$, $z = 2$, $z = 3$. Geometrically, they are obtained by cutting the surface (see the last figure

Notes:

in Figure 5.19) using horizontal planes that parallel the xy-plane. These planes are of the form $z = z_k$ and the resulting curves are called **contour curves**. When these contour curves are projected on the xy-plane, the resulting curves in the xy-plane are called **level curves**. See Figure 5.19 (a). In the case of the example 5.3.3, the level curves are shown in 5.19 (b).

Example 5.3.5

Find and sketch the level curves of each of the following surfaces.

a) $z = x + y$.

b) $z = x^2 - y^2$.

c) $z = 2^{x^2-y}$.

Solution

a) Level curves:

$$z = 0 \Rightarrow x+y = 0,\ z = \pm 1 \Rightarrow x+y = \pm 1,\ z = \pm 2 \Rightarrow x+y = \pm 2, ...$$

Keep doing this for some more values of z. We can see then that they are lines in the xy-plane. See Figure 5.21 (a).

b) Level curves:

$$z = 0 \Rightarrow x^2 - y^2 = 0 \Rightarrow y^2 = x^2 \Rightarrow y = \pm x \Rightarrow \text{ pair of lines }.$$

$$z = 1 \Rightarrow x^2 - y^2 = 1 \Rightarrow \text{ sideway hyperbolas, but if}$$
$$z = -1 \Rightarrow y^2 - x^2 = 1 \Rightarrow \text{ up-down hyperbolas.}$$

Likewise, for

$$z = 4 \Rightarrow x^2 - y^2 = 4 \Rightarrow \frac{x^2}{2^2} - \frac{y^2}{2^2} = 1 \Rightarrow \text{ sideway hyperbolas.}$$

$$z = -4 \Rightarrow x^2 - y^2 = -4 \Rightarrow \frac{y^2}{2^2} - \frac{x^2}{2^2} = 1 \Rightarrow \text{ up-down hyperbolas.}$$

See Figure 5.21 (b).

c) Level curves:

$$z = 1 \Rightarrow 2^{x^2-y} = 1 \Rightarrow x^2 - y = 0 \Rightarrow y = x^2.$$

$$z = 2 \Rightarrow 2^{x^2-y} = 2 \Rightarrow x^2 - y = 1 \Rightarrow y = x^2 - 1.$$

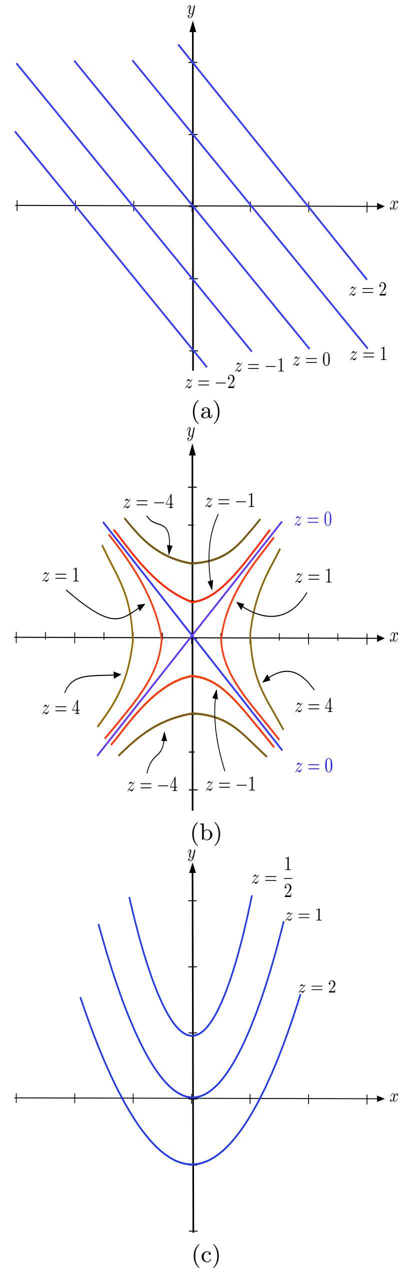


Figure 5.21: Example 5.3.5

Notes:

$$z = \frac{1}{2} \Rightarrow 2^{x^2-y} = \frac{1}{2} = 2^{-1} \Rightarrow x^2 - y = -1 \Rightarrow y = x^2 + 1.$$

Keep doing this for some more values of z. We can see then that they are parabolas in the xy-plane. See Figure 5.21 (c).

Example 5.3.6

In their original paper, Cobb and Douglas estimated the production function for the United States to be $z = 1.01x^{3/4}y^{1/4}$, where x represents the amount of labour and y represents the amount of capital. Now, this is known as the Cobb-Douglas production type of function. Find and sketch the level curve at a production of 500 of this function.

Solution When $z = 500 \Rightarrow 500 = 1.01x^{3/4}y^{1/4} \Rightarrow y^{1/4} = \dfrac{500}{1.01x^{3/4}} \Rightarrow y = \left(\dfrac{500}{1.01}\right)^4 \dfrac{1}{x^3} \Rightarrow y \approx \dfrac{6 \cdot 10^{10}}{x^3}$.

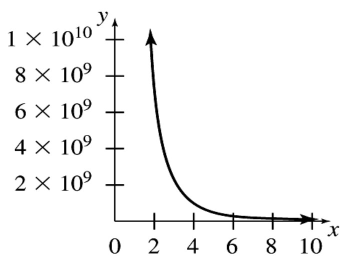


Figure 5.22

Example 5.3.7

Let $f(x, y) = 100x^{2/3}y^{1/3},\ x > 0,\ y > 0$. Find the level curve of f that passes through the point $(8, 27)$.

Solution Level curve is $z_0 = f$, where $z_0 = f(8, 27)$. So, at this point, we have $100x^{2/3}y^{1/3} = 100(8)^{2/3}(27)^{1/3} = 100 \cdot 4 \cdot 3 \Rightarrow x^{2/3}y^{1/3} = 12 \Rightarrow y^{1/3} = \dfrac{12}{x^{2/3}} \Rightarrow y = \dfrac{12^3}{x^2}$.

Note: In general, the Cobb-Douglas production function has the form:

$$P(K, L) = A \cdot K^{\alpha} \cdot L^{1-\alpha}, A \text{ is a constant, and } 0 < \alpha < 1.$$

Here, $K = \#$ of units of capital and $L = \#$ of units of labour are used in the manufacturing process. $P = \#$ of units of production.

Notes:

Exercises 5.3

Problems

1. The original Cobb-Douglas production function was $z = 1.01x^{3/4}y^{1/4}$, where x represented units of labour (in hours labour) and y of capital (in dollars cost of replacement) used in the production of z units of output (in dollars).

   a) Find the formula (y as a function of x) for the level curves of the function when production is $z = 5.05$.

   b) Confirm that $x = 5$ and $y = 5$ is on this iso-product curve; and find another point on this curve (preferably with integral or finite-decimal representation).

   c) Sketch the iso-product curve on an xy-axis system.

In Exercises 2 – 6, for each function sketch the contours for the given z-values on a single xy-axis system.

2. $f(x, y) = (2y - x)^2; z = 0, 1, 2.$

3. $f(x, y) = y^{1/2} - x^{1/2}; z = 0, 1, 2.$

4. $f(x, y) = x^2 - y; z = 0, 1, 2.$

5. $f(x, y) = xe^y; z = 1, -1, e, -e.$

6. $f(x, y) = \ln\left(x^2 + \frac{y^2}{9}\right); z = 0, 1, \ln 4.$

In Exercises 7 – 11, sketch the following traces on xz-axes or yz-axes.

7. $f(x, y) = (2y - x)^2; x = 0, 1; y = 0, \frac{1}{2}.$

8. $f(x, y) = y^{1/2} - x^{1/2}; x = 0, 1; y = 0, 1.$

9. $f(x, y) = x^2 - y; x = 0, 1, -2; y = 0, 1, -2.$

10. $f(x, y) = xe^y; x = 0, 1, -1; y = 0, \ln 2, -\ln 2\Big(= \ln(1/2)\Big).$

11. $f(x, y) = \ln\left(x^2 + \frac{y^2}{9}\right); x = 0, 1, \frac{1}{3}; y = 0, 1, 3.$

12. Consider the Cobb-Douglas production function $z = f(x, y) = 100x^{2/3}y^{1/3}$. Sketch the following traces on yz-axes or xz-axes: $x = 0, 1, 8; y = 0, 1, 8.$

.

5.4 Partial Derivatives

In this section, we study how the derivatives of multivariable functions are found and interpreted.

Suppose that a company makes only tables and chairs. The profits of the company are given $P = f(x, y) = 4x^2 - 5xy + 5y^2 - 40$, where x is the number of tables, and y is the number of chairs made and sold. Suppose that sales of tables have been steady at 10 units, while the sales of chairs vary. How will a change in y affect P?

To answer this question, we look at the marginal profit with respect to y. Recall the marginal profit is the derivative of the profit function. Here, $x = 10$ is fixed. Let $P(y) = f(10, y)$, that is, $P(y) = 4(10)^2 - 5(10)y + 5y^2 - 40 \Rightarrow P(y) = 360 - 50y + 5y^2$. This function shows the profits from the sales of y chairs, assuming that x is fixed at 10 units. The marginal profit is then $\dfrac{dP}{dy} = -50+10y$. That is, the derivative of $f(x, y)$ with respect to y (only) is $-50 + 10y$, assuming that x is fixed. This example above leads to so called **partial derivatives** . A definition of partial derivatives of f respect to x and y is as follows.

Definition 5.4.1 Partial Derivatives

Let $z = f(x, y)$ be a function of two independent variables. Let all the indicated limits exist. Then, the partial derivatives of f with respect to x, denoted by $f_x(x, y)$ or $\dfrac{\partial f}{\partial x}$, and with respect to y, denoted by $f_y(x, y)$ or $\dfrac{\partial f}{\partial y}$, are defined by

$$f_x(x, y) = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h},$$

and

$$f_y(x, y) = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

If the limits do not exist, then the partial derivatives do not exist.

Note: When we do derivatives with usually use rules of differentiation.

Notes:

These rules can be applied to partial derivatives too. In practice, we rarely use the definition above. So, unless we are asked to use the formal limit definition of partial derivatives, we compute partial derivatives by applying those rules that we have learned from Calculus I to function $z = f(x, y)$ keeping one variable fixed at a time.

Example 5.4.1

Find partial derivatives of each of the following functions.

a) $f(x, y) = \frac{1}{6}x^2y^3 + \frac{1}{20}x^5y^4$.

b) $g(x, y) = \ln\left|x^3 + 3y^2\right|$.

c) $h(x, y) = e^{x^2y}$.

Solution

a) $f(x, y) = \frac{1}{6}x^2y^3 + \frac{1}{20}x^5y^4$.

$$f_x(x, y) = \frac{1}{3}xy^3 + \frac{1}{4}x^4y^4, \quad f_y(x, y) = \frac{1}{2}x^2y^2 + \frac{1}{5}x^5y^3.$$

b) $g(x, y) = \ln\left|x^3 + 3y^2\right|$.

$$g_x(x, y) = \frac{1}{x^3 + 3y^2}\cdot(3x^2) = \frac{3x^2}{x^3 + 3y^2}, \quad g_y(x, y) = \frac{1}{x^3 + 3y^2}\cdot(6y) = \frac{6y}{x^3 + 3y^2}.$$

c) $h(x, y) = e^{x^2y}$.

$$h_x(x, y) = e^{x^2y}\cdot(2xy) = 2xye^{x^2y}, \quad h_y(x, y) = e^{x^2y}\cdot(x^2) = x^2e^{x^2y}.$$

Notes:

Notation for First Partial Derivatives:

- The first partial derivatives of $z = f(x, y)$ are denoted by
$$\frac{\partial z}{\partial x} = f_x(x, y) = z_x = \frac{\partial}{\partial x}\Big[f(x, y)\Big], \text{ and } \frac{\partial z}{\partial y} = f_y(x, y) = z_y = \frac{\partial}{\partial y}\Big[f(x, y)\Big].$$

- The values of the first partial derivatives at the point (a, b) are denoted by
$$\left.\frac{\partial z}{\partial x}\right|_{(a,b)} = f_x(a, b), \text{ and } \left.\frac{\partial z}{\partial y}\right|_{(a,b)} = f_y(a, b).$$

Example 5.4.2
Find the first partial derivatives of $z = f(x, y) = xe^{x^2 y}$ at the point $(-1, 1)$.

Solution

$$f_x(x, y) = (1)e^{x^2 y} + xe^{x^2 y} \cdot 2xy = (1 + 2x^2 y)e^{x^2 y} \Rightarrow f_x(-1, 1) = 3e,$$

and

$$f_y(x, y) = (0)e^{x^2 y} + xe^{x^2 y} \cdot x^2 = x^3 e^{x^2 y} \Rightarrow f_y(-1, 1) = -e,$$

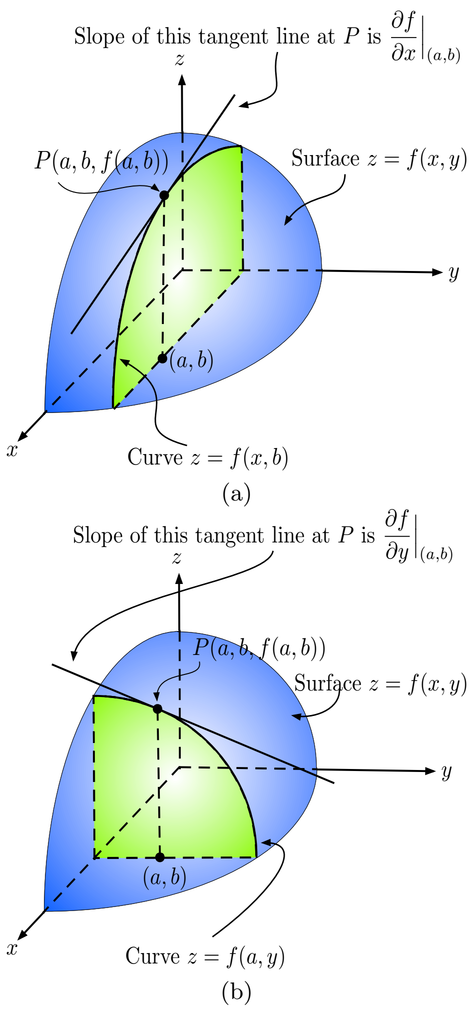


Figure 5.23: Graphical Interpretation of Partial Derivatives.

Graphical Interpretation of Partial Derivatives: .
Consider the function $z = f(x, y)$, and a point (a, b) on the xy-plane. From the figures above, we can see that the graph of this function is a surface in space.

If the variable y is fixed, say, $y = b$, then $z = f(x, b)$ is a function of one variable, x. The graph of this function is the curve that is the intersection of the plane $y = b$, and the surface $z = f(x, y)$. On this curve, the partial derivative $f_x(x, b)$ represents the slope in the plane $y = b$, or the slope of the tangent line to the curve at (a, b).

Similarly, if we keep the variable x fixed, say, $x = a$, then $z = f(a, y)$ is a function of one variable, y, whose graph is the intersection of the plane $x = a$ and the surface $z = f(x, y)$. On this curve, the partial derivative $f_y(a, y)$ represents the slope in the plane $x = a$, or the slope of the tangent

Notes:

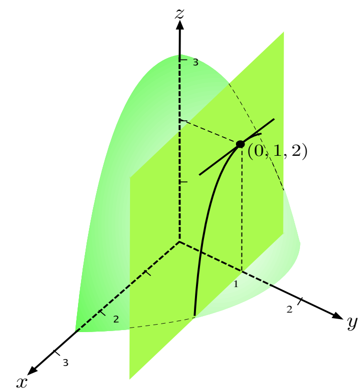
line to the curve at (a, b).

So, the partial derivative $f_x(a,b) = \left.\dfrac{\partial f}{\partial x}\right|_{(a,b)}$ is the slope in the x-direction of the tangent line or surface at the point $P(a, b, f(a, b))$. Likewise, the partial derivative $f_y(a,b) = \left.\dfrac{\partial f}{\partial y}\right|_{(a,b)}$ is the slope in the y-direction of the tangent line or surface at the point $P(a, b, f(a, b))$.

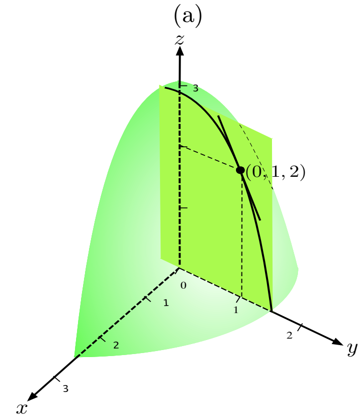
Example 5.4.3

Find the slopes of the surface given by $z = f(x,y) = -\dfrac{12}{25}x^2 - \dfrac{4}{3}y^2 + \dfrac{10}{3}$ at the point $(0, 1, 2)$ in both x and y directions.

Solution $f_x(x,y) = -\dfrac{24}{25}x \Rightarrow f_x(0,1) = 0$, and $f_y(x,y) = -\dfrac{8}{3}y \Rightarrow f_y(0,1) = -\dfrac{8}{3}$. See Figure 5.24.



(a)



(b)

Figure 5.24: Example 5.4.3

Example 5.4.4

The cost of making x tables and y chairs is given by $C(x,y) = 200 + 10x + 4y - 4\sqrt{xy}$.

a) Find the fixed cost and cost when 10 tables and 40 chairs are made.

b) Find marginal cost of making the tables and chairs when 10 tables and 40 chairs are produced.

Solution

a) The fixed cost is $F = f(0,0) = 200$ dollars. $C(10, 40) = 200 + 100 + 160 - 80 = 380$ dollars.

b) The marginal cost of making tables when 10 tables and 40 chairs are produced is

$$C_x(x,y) = 10 - 4\frac{\sqrt{y}}{2\sqrt{x}} \Rightarrow C_x(10,40) = 10 - 2\frac{\sqrt{40}}{\sqrt{10}} = 6 \text{ dollars.}$$

The marginal cost of making chairs when 10 tables and 40 chairs are produced is

$$C_y(x,y) = 10 - 4\frac{\sqrt{x}}{2\sqrt{y}} \Rightarrow C_y(10,40) = 4 - 2\frac{\sqrt{10}}{\sqrt{40}} = 3 \text{ dollars.}$$

Notes:

Example 5.4.5

The Cobb-Douglas production function is given by $P(K,L) = 120K^{1/3}L^{2/3}$. When 27 units of capital and 8 units of labour are being used,

a) how much would the production increase with additional unit of capital?

b) how much would the production increase with additional unit of labour?

Solution

a) how much would the production increase with additional unit of capital?

$$\frac{\partial P}{\partial K}(x,y) = 40K^{-2/3}L^{2/3} \Rightarrow \left.\frac{\partial P}{\partial K}\right|_{(27,8)} \approx 17.8.$$

So, the production would increase by about 18 units.

b) how much would the production increase with additional unit of labour?

$$\frac{\partial P}{\partial L}(x,y) = 80K^{1/3}L^{-1/3} \Rightarrow \left.\frac{\partial P}{\partial L}\right|_{(27,8)} = 120.$$

So, the production would increase by about 120 units.

Example 5.4.6

A company that manufactures computers has determined that its production function is given by $P(x,y) = 0.1xy^2 \ln(2x+3y+2)$, where x is the size of the labour force measured in work-hours per week, and y is the amount of capital measured in in units of \$1000 invested. Find the marginal productivity of labour and capital when $x = 50$ and $y = 20$, and then interpret the results.

Solution

$$P_x(x,y) = 0.1\left[y^2 \ln(2x+3y+2) + \frac{x^2y}{2x+3y+2}\cdot(2)\right] \Rightarrow P_x(50,20) \approx 228.$$

This means that if the capital investment is held constant at \$20,000 and the labour is increased from 50 to 51 work-hours per week, then the production will increase by about 228 units.

$$P_y(x,y) = 0.1\left[2xy \ln(2x+3y+2) + \frac{x^2y}{2x+3y+2}\cdot(3)\right] \Rightarrow P_y(50,20) \approx 1055.$$

Notes:

This means that if the work-hours is held constant at 50 hours per week and the capital investment is increased from \$20,000 to \$21,000, then the production will increase by about 1055 units.

Partial Derivatives of Functions of Three Variables:

The idea is the same, but instead of keeping one variable fixed, this time we keep two variables fixed while taking derivative with respect to the other variable. The notations are still the same. Let $w = f(x, y, z)$. Partial derivatives with respect to x, y, and z, respectively, of w are

$$\frac{\partial w}{\partial x} = f_x(x, y, z),\ \frac{\partial w}{\partial y} = f_y(x, y, z),\ \frac{\partial w}{\partial z} = f_z(x, y, z).$$

Example 5.4.7

Suppose that the temperature at any point (x, y, z) in space is given by $T(x, y, z) = \dfrac{100}{1 + \sqrt{x^2 + y^2 + z^2}}$. What is the rate of change of the temperature at the point $P(1, 2, 2)$, with respect to the distance in the direction of straight up?

Solution $\dfrac{\partial T}{\partial z} = -100 \cdot \left(1 + \sqrt{x^2 + y^2 + z^2}\right)^{-2} \cdot \dfrac{1}{2\sqrt{x^2 + y^2 + z^2}} \cdot (2z) \Rightarrow \left.\dfrac{\partial T}{\partial z}\right|_{(1,2,2)} = -4.125$ degrees per unit distance.

Example 5.4.8

Let A be the amount, in dollars, after t years of a principal P dollars invested at interest rate r per year compounded continuously. It is governed by the formula $A = Pe^{rt}$. As we can see it is a function of P, r, and t. It is linear in P and exponential in r and t. So, we can write $A = f(P, r, t) = Pe^{rt}$. Suppose that $P = 100$ dollars, $r = 10\%$, and $t = 3$. Find $\left.\dfrac{\partial A}{\partial t}\right|_{(100,0.10,3)}$. What does it mean?

Solution $\dfrac{\partial A}{\partial t} = Pe^{rt} \cdot r = Pre^{rt} \Rightarrow \left.\dfrac{\partial A}{\partial t}\right|_{(100,0.10,3)} = 100(0.10)e^{0.10 \cdot 3} =$ 134.99, (dollars/year). This means that if t is increased by 1 year (from $t = 3$ to $t = 4$), then the amount A will be increased by about \$135.

Notes:

Higher-Order Partial Derivatives:

Just like ordinary derivatives, we can take second, third, and higher partial derivatives of a function of several variables as well, provided that those derivatives exist. In this class, we concentrate on the second derivatives of functions of two independent variables only. Let $z = f(x, y)$.

- Differentiate twice with respect to x: $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right) = \dfrac{\partial^2 z}{\partial x^2} = f_{xx}$.
- Differentiate twice with respect to y: $\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial^2 z}{\partial y^2} = f_{yy}$.
- Differentiate first with respect to x, and then to y: $\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right) = \dfrac{\partial^2 z}{\partial y \partial x} = \Big(f_x\Big)_y = f_{xy}$.
- Differentiate first with respect to y, and then to x: $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial^2 z}{\partial x \partial y} = \Big(f_y\Big)_x = f_{yx}$.

Note: The last two $f_{xy}(x, y)$ and $f_{yx}(x, y)$ are called mixed partial derivatives.

Example 5.4.9
Suppose that $z = f(x, y) = 2x^3 - 4x^2y + 3y^3$. Find the second partial derivatives and determine the value of $f_{xy}(-1, 2)$.

Solution First we find the first order partial derivatives of f.

$$f_x(x, y) = 6x^2 - 8xy, \quad f_y(x, y) = -4x^2 + 9y^2.$$

Now, we find second order derivatives of f.

$$f_{xx}(x, y) = 12x - 8y, \quad f_{yy}(x, y) = 18y, \quad f_{xy}(x, y) = -8x, \quad f_{yx}(x, y) = -8x$$

$$\Rightarrow f_{xy}(-1, 2) = 8.$$

Note: Under usual conditions, we have $f_{xy}(x, y) = f_{yx}(x, y)$.

Notes:

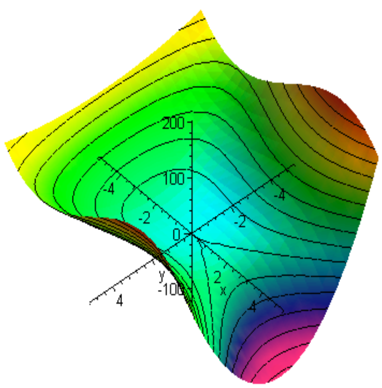
Exercises 5.4

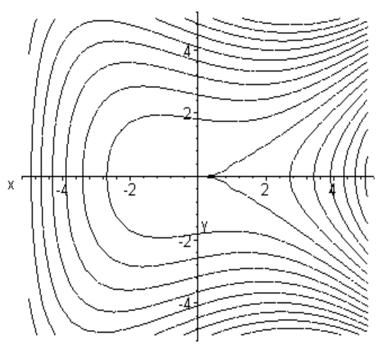
Problems

1. Suppose that $z = f(x,y) = 100x^{2/3}y^{1/3}$, where x represented units labour and y units of capital, and z was units of production. What is the interpretation of $\frac{\partial}{\partial x}f(x,y) = f_x(x,y)$? (HINT: the **marginal productivity of labour**). Give its mathematical meaning and its units of measure; and evaluate and interpret when $x = 8$ and $y = 27$.

2. Repeat equation 1 for $\frac{\partial}{\partial y}f(x,y) = f_y(x,y)$.

In Exercises 3 – 4, the 3-D plot of the surface of a function f (perversely, MAPLE has produced a left handed 3-D coordinate system - the usual convention is to have a right-handed system: as shown, the positive x-axis extends downward to the right, the positive y-axis is extends downward to the left); and its contour plot are shown below.





[For those interested, the function plotted is: $f(x,y) = x^2y + 6x^2 - y^3$.]
For exercises 3 and 4, give the sign (positive, negative or 0) of the partial derivatives listed for f whose graph is shown.

3. a) $f_x(0,0)$;
   b) $f_y(0,0)$;
   c) $f_x(-2,0)$;
   d) $f_y(-2,0)$.

4. a) $f_x(0,-2)$;
   b) $f_y(0,-2)$;
   c) $f_x(2,0)$;
   d) $f_x(2,0)$.

In Exercises 5 – 12, find f_x, f_y, f_{xx}, f_{xy} and f_{yy}. Then compute $D = f_{xx} \cdot f_{yy} - (f_{xy})^2$ (this is called the Hessian and will have important use in the next section), simplifying as much as possible.

5. $f(x,y) = 100 - 36x^2 - 64y^2$.

6. $f(x,y) = (100 - 36x^2 - 64y^2)^{1/2}$.

7. $f(x,y) = y \ln x$.

8. $f(x,y) = y^x$.

9. $f(x,y) = ye^{x+y}$.

10. $f(x,y) = \dfrac{x+y}{x^2+y^2}$.

11. $f(x,y) = \arctan(x/y)$.

In Exercises 12 – 15, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. Evaluate at $x = y = 1$ for $f(1) = 2$, $f'(1) = 5$, and $f''(1) = -3$.

12. $z = f(x/y)$.

13. $z = f(x \cdot y)$, and find the Hessian D defined above.

14. $z = f(x)f(y)$, and find the Hessian D defined above.

15. For the Cobb-Douglas function $f(x,y) = ax^{\alpha}y^{\beta}$, where $a > 0$, and $\alpha, \beta > 0$.
    a) Find the second order partial derivatives of f: f_{xx}, f_{xy} and f_{yy} (and f_{yx} as a check on f_{xy}, if you like).
    b) Then find the Hessian $D = f_{xx} \cdot f_{yy} - (f_{xy})^2$ as defined above. On simplifying your expression for the Hessian, you should find it reduces to $D(x,y) = a^2x^{2(\alpha-1)}y^{2(\beta-1)}\{1 - (\alpha+\beta)\}$.

5.5 Maxima and Minima

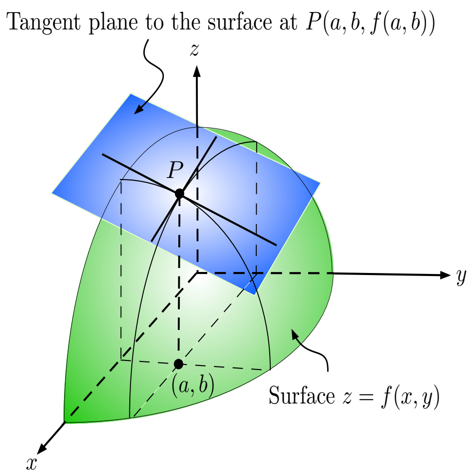


Figure 5.25: Tangent plane for the curve $z = f(x,y)$.

Tangent Planes for the Surface $z = f(x,y)$:

Let f be differentiable at the point (a,b). That is, f is a function, whose partial derivatives exist at the (a,b). An equation of the plane tangent to the surface $z = f(x,y)$ at the point $(a,b,f(a,b))$ is

$$z = f_x(a,b) \cdot (x-a) + f_y(a,b) \cdot (y-b) + f(a,b).$$

See Figure 5.25.

Example 5.5.1
Find an equation of the plane tangent to the paraboloid $z = f(x,y) = 32 - 3x^2 - 4y^2$ at $(2,1,16)$.

Solution $f_x(x,y) = -6x$ and $f_y(x,y) = -8y$. Then, $f_x(2,1) = -12$ and $f_y(2,1) = -8$. So, the equation of the plane tangent to the curve is

$$z = f_x(a,b) \cdot (x-a) + f_y(a,b) \cdot (y-b) + f(a,b) = -12(x-2) - 8(y-1) + 16$$

$$\Rightarrow z = -12x - 8y + 48.$$

As we can see from Figure 5.25., the tangent plane at $P(a,b,f(a,b))$ form by two tangent lines to the curves $z = f(a,y)$ and $z = f(x,b)$. These two lines have slopes $f_y(a,b)$ and $f_x(a,b)$ respectively. If these lines are horizontal, we have horizontal tangent plane. This implies that $\left.\frac{\partial z}{\partial x}\right|_{(a,b)} = 0$ and $\left.\frac{\partial z}{\partial y}\right|_{(a,b)} = 0$. Then, the point $(a,b,f(a,b))$ is called **critical point**. We'll talk about critical points later.

Recall linear approximation. Near (a,b), we can approximate the value of the function f at (x,y) using

$$f(x,y) \approx f(a,b) + f_x(a,b) \cdot (x-a) + f_y(a,b) \cdot (y-b).$$

Example 5.5.2
Approximate $f(2.1, 1.1)$ near $(2,1)$ for the function in Example 1.

Notes:

Solution $f(2.1, 1.1) \approx f_x(2,1)\cdot(2.1-2) + f_y(2,1)\cdot(1.1-1) + f(2,1) = -12(0.1) - 8(0.1) + 16 = -0.1\cdot 20 + 16 = -2 + 16 = 14.$

Relative (or Local) Maxima and Minima:

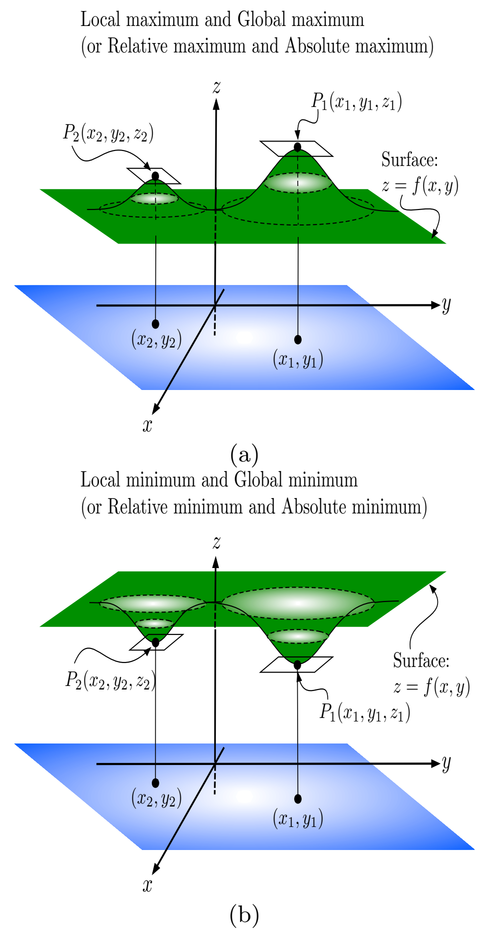


Figure 5.26

Definition 5.5.1 Relative or Local Extrema

Let f be a function defined on a region containing (a, b).

1. The function f has a relative or local maximum at (a, b) if there is a circular region centred at (a, b) such that $f(x, y) \leq f(a, b)$, for all (x, y) in the region.

2. The function f has a relative or local minimum at (a, b) if there is a circular region centred at (a, b) such that $f(x, y) \geq f(a, b)$, for all (x, y) in the region.

Definition 5.5.2 Absolute or Global Extrema

Let f be a function defined on a region containing (a, b).

1. If $f(x, y) \leq f(a, b)$, for all (x, y) in the domain of f, then f has a global maximum value at (a, b).

2. If $f(x, y) \geq f(a, b)$, for all (x, y) in the domain of f, then f has a global minimum value at (a, b).

In Figure 5.26 (a), the function $z = f(x, y)$ has local maxima at both (x_1, y_1) and (x_2, y_2), but it has global maximum at only (x_1, y_1).

In Figure 5.26 (b), the function $z = f(x, y)$ has local minima at both (x_1, y_1) and (x_2, y_2), but it has global minimum at only (x_1, y_1).

From Figure **??** and Figure **??**, it seems that the tangent planes at both P_1 and P_2 are both horizontal. It means that $\left.\dfrac{\partial z}{\partial x}\right|_{(x_k, y_k)} = 0$ and $\left.\dfrac{\partial z}{\partial y}\right|_{(x_k, x_k)} = 0$, for $k = 1, 2$. This in turn means that at the maximum and minimum points the partial derivatives are zero. So, we have the following theorem

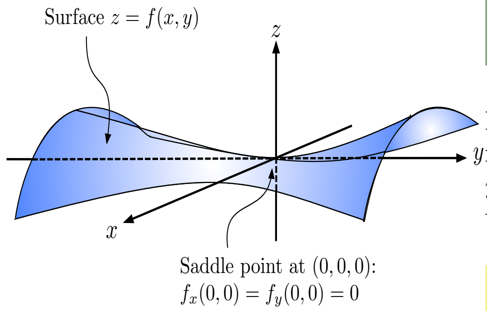
Notes:

Theorem 5.5.1 Derivatives and Local Extrema

Let a function $z = f(a, b)$ have a relative or local maximum or minimum at the point $(a, b, f(a, b))$. Let $\left.\dfrac{\partial z}{\partial x}\right|_{(a,b)}$ and $\left.\dfrac{\partial z}{\partial y}\right|_{(a,b)}$ both exist. Then,

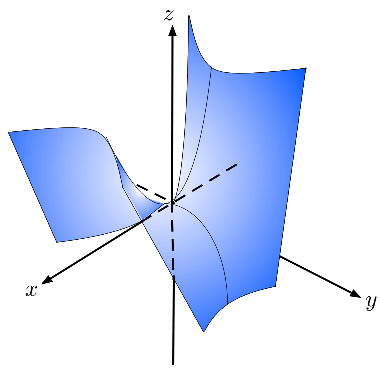
$$\left.\frac{\partial z}{\partial x}\right|_{(a,b)} = 0 \text{ and } \left.\frac{\partial z}{\partial y}\right|_{(a,b)} = 0.$$

Notice that this theorem is known as the first partial derivative test for relative extrema. However, it involves study of signs on both the x and y directions of the function around the point. It is complicated. We'll do the test for extrema later using second partial derivative test.



(a)

Hyperbolic paraboloid: $z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$



(b)

Figure 5.27: Saddle Points.

Definition 5.5.3 Critical Number and Critical Point

An interior point (a, b) in the domain of f is a critical number of f if either

$f_x(a, b) = f_y(a, b) = 0$, or one (or both) f_x or f_y does not exist at (a, b).

Then, the critical point is $(a, b, f(a, b))$. The critical points are candidates for local maximum and minimum.

Note: The fact that the slopes of the tangent lines are zero (that is $f_x(a, b) = f_y(a, b) = 0$) at a point does not mean that a local extremum occurs at that point. For instance, see Figure 5.27 (a).

As we can see from the graph in Figure 5.27 (a), $f_x(0, 0) = f_y(0, 0) = 0$, but the surface rises from $(0, 0)$ along the y-axis, but falls from $(0, 0)$ along the x-axis. It does not satisfy both the definitions of the maximum and minimum. So, the function f does not have any extrema at the origin. This type of point is called **saddle point**. The same for this hyperbolic paraboloid in Figure 5.27 (b). It does not have a local maximum/minimum at the origin.

Example 5.5.3

Find the critical point(s) of each of the following functions, if any.

a) $f(x, y) = 2x^2 + y^2 + 8x - 6y + 20.$

b) $f(x, y) = 3 - \sqrt[3]{x^2 + y^2}$

Notes:

c) $g(x, y) = xy + \dfrac{1}{x} + \dfrac{1}{y}$, $x > 0$, $y > 0$.

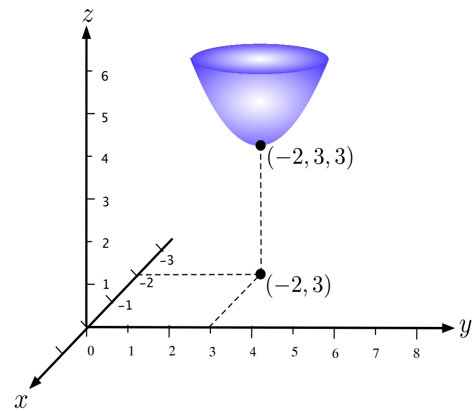
Solution

a) $f_x(x, y) = 4x + 8 = 0 \Rightarrow x = -2$ and $f_y(x, y) = 2y - 6 = 0 \Rightarrow y = 3$. So, the critical number is $(-2, 3)$. Then, substituting these values into the f gives $f(-2, 3) = 3$. Thus, a critical point is $(-2, 3, 3)$. Notice that the graph of the function f shows that it has a relative minimum there. See Figure 5.28 (a).

b) $f_x(x, y) = -\dfrac{2x}{3(x^2 + y^2)^{2/3}}$, $f_y(x, y) = -\dfrac{2y}{3(x^2 + y^2)^{2/3}}$. These partial derivatives are defined for all points in the xy-plane except at $(0, 0)$. So, it is a critical number of f. Also, $f(0, 0) = 3$. Thus, the critical point is $(0, 0, 3)$. Notice that the graph of the function f shows that it has a relative maximum there. See Figure 5.28 (b).

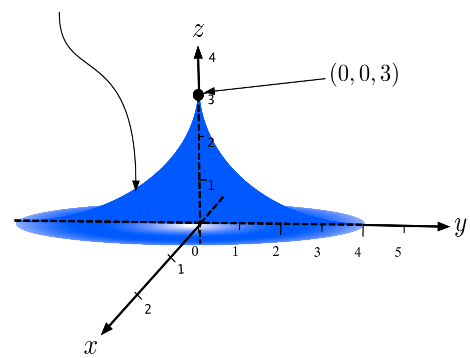
c) $g_x(x, y) = y - \dfrac{1}{x^2}$, and $g_y(x, y) = x - \dfrac{1}{y^2}$. For x, $y > 0$, these two derivatives are zero when $y = \dfrac{1}{x^2}$, and $x = \dfrac{1}{y^2}$. Solving these two equation, we obtain $x - x^4 = 0 \Rightarrow x(1 - x^3) = 0 \Rightarrow x = 0$ or $x = 1$. But, $x > 0$, we take $x = 1$ only. Then, $y = 1$ as well. $g(1, 1) = 3$. So, the critical point is $(1, 1, 3)$.

Surface: $f(x, y) = 2x^2 + y^2 + 8x - 6y + 20$



(a)

Surface: $f(x, y) = 3 - \sqrt[3]{x^2 + y^2}$



$f_x(x, y)$ and $f_y(x, y)$ are underfined at $(0, 0)$.

(b)

Figure 5.28: Example 5.5.3

The results of the following theorem can be used to find out whether a critical point is a maximum or a minimum.

Notes:

Theorem 5.5.2 Test for Local Extrema

For a function $z = f(x, y)$, let $f_{xx},\ f_{xy},\ f_{yy}$ all exist in an open region in the xy-plane containing (a, b) for which $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Let D be defined by

$$D = f_{xx}(a, b) \cdot f_{yy}(a, b) - \Big[f_{xy}(a, b)\Big]^2.$$

Then,

1. If $D > 0$ and $f_{xx}(a, b) > 0$, then f has a local min at (a, b).
2. If $D > 0$ and $f_{xx}(a, b) < 0$, then f has a local max at (a, b).
3. If $D < 0$, then the point $(a, b, f(a, b))$ is a saddle point.
4. If $D = 0$, the test is inconclusive (it gives no information).

Example 5.5.4
Given the function $z = f(x, y) = x^3 + x(y^2 - 1)$, find and classify all the critical points.

Solution $f_x = 3x^2 + y^2 - 1 = 0$ and $f_y = 2xy = 0 \Rightarrow x = 0$ or $y = 0$. If $x = 0$, the first equation gives $y = \pm 1$. If $y = 0$, the first equation gives $x = \pm\dfrac{1}{\sqrt{3}}$. So, we have four critical points at $(0, \pm 1)$, and $\Big(\pm\dfrac{1}{\sqrt{3}}, 0\Big)$. Now,

$f_{xx} = 6x$, $f_{yy} = 2x$ and $f_{xy} = 2y$. Then, $D = f_{xx} \cdot f_{yy} - (f_{xy})^2 = 12x^2 - 4y^2$.
$D(0, \pm 1) = -4 < 0$, we have saddle points at $(0, \pm 1)$. $D\Big(\pm\dfrac{1}{\sqrt{3}}, 0\Big) = 4 > 0$, we know that local extremum occur at these points. Since $f_{xx}\Big(\dfrac{1}{\sqrt{3}}, 0\Big) = \dfrac{6}{\sqrt{3}} > 0$, f has local minimum at $\Big(\dfrac{1}{\sqrt{3}}, 0\Big)$. Since $f_{xx}\Big(-\dfrac{1}{\sqrt{3}}, 0\Big) = -\dfrac{6}{\sqrt{3}} < 0$, f has local maximum at $\Big(-\dfrac{1}{\sqrt{3}}, 0\Big)$.

Example 5.5.5
Given the function $z = f(x, y) = xye^{-(x^2+y^2)/2}$, find and classify all the critical points.

Notes:

Solution $f_x = y(1-x^2)e^{-(x^2+y^2)/2} = 0 \Rightarrow y(1-x^2) = 0$ and $f_y = x(1-y^2)e^{-(x^2+y^2)/2} = 0 \Rightarrow x(1-y^2) = 0$. Solving these two equations gives $x = 0,\ y = \pm 1$ and $y = 0,\ x = \pm 1$. So, there are five critical numbers $(0,0)$, $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$. Now,

$$f_{xx} = -xy(3-x^2)e^{-\frac{x^2+y^2}{2}},\ f_{yy} = -xy(3-y^2)e^{-\frac{x^2+y^2}{2}}, f_{xy} = (1-x^2)(1-y^2)e^{-\frac{x^2+y^2}{2}}.$$

$$D(x,y) = \Big[(xy)^2(3-x^2)(3-y^2) - (1-x^2)^2(1-y^2)^2\Big]e^{-\frac{x^2+y^2}{2}}.$$

Testing all those critical numbers, we get a saddle point at $(0,0)$, local minimum at $(1,-1)$ and $(-1,1)$, and local maximum at $(1,1)$ and $(-1,-1)$.

For global extrema, it is a bit complicated. It is possible that f has a unique critical point at (a,b) at where $f(a,b)$ is a local extremum, but it is not a global extremum. However, some production polynomial functions in business or any function of x and y, whose graph is a quadric surface, then any local extremum is automatically a global extremum.

Example 5.5.6

Let $f(x,y) = 3xe^y - x^3 - e^{3y}$. Find global extrema, if any.

Solution Domain of f is the set of all (x,y).

$$f_x = 3e^y - 3x^2 = 0, \text{ and } f_y = 3xe^y - 3e^{3y} = 0.$$

The first equation gives $e^y = x^2$. Sub. into the second equation gives $3xe^y - \Big(e^y\Big)^3 = 0 \Rightarrow x(x^2) - (x^2)^3 = 0 \Rightarrow x^3 - x^6 = 0 \Rightarrow x^3(1-x^3) = 0 \Rightarrow x = 0$ or $x = 1$. Since $e^y = x^2$, we reject $x = 0$. So, we have only one critical number, which is $(1,0)$ because if $x = 1$, $e^y = 1 \Rightarrow y = 0$. Now, Keeping $y = 0$, $f(x,0) = 3x - x^3 - 1$. $f(1,0) = 1$, but $\lim\limits_{x\to-\infty} f(x,0) = \infty$ and $\lim\limits_{x\to\infty} f(x,0) = -\infty$. So, there is no global extremum.

Example 5.5.7

Let P be a production function given by

$$P(L,K) = 0.54L^2 - 0.02L^3 + 1.89K^2 - 0.09K^3,$$

where L and K are the amounts of labour and capital, respectively, and P is the quantity of output produced. Find the values of L and K so that P is maximized.

Solution First, we find critical numbers.

$$P_L = 1.08L - 0.06L^2 = 0.06L(18-L) = 0,$$

Notes:

$$P_K = 3.78K - 0.27K^2 = 0.27K(14 - K) = 0.$$

Solving these two gives $L = 0,\ 18$ and $K = 0,\ 14$. There are four critical numbers are: $(0,0)$, $(0,14)$, $(18,0)$, and $(18,14)$.

$$P_{LL} = 1.08 - 012L,\ \ P_{KK} = 3.78 - 0.54K,\ \ P_{LK} = 0$$

$$\Rightarrow D(L,K) = (1.08 - 0.12L)(3.78 - 0.54K).$$

At $(0,0)$, $D(0,0) = 1.08(3.78) > 0$, and $P_{LL}(0,0) = 1.08 > 0$, there is a local minimum at $(0,0)$. At $(0,14)$, $D(0,14) = 1.08(-3.78) < 0$, and $P_{LL}(0,14) = 1.08 > 0$, there is no local extrema there. Likewise, there is no local extrema at $(18,0)$. Now, at $(18,14)$, $D(18,16) = (-1.08)(-3.78) > 0$, but $P_{LL}(18,14) = -1.08 < 0$, there is a local maximum at $(18,14)$. Therefore, the maximum output is obtained when $L = 18$ and $K = 14$.

Example 5.5.8

The demand functions for two products are given by

$$x_1 = 100 - 2p_1 + 1.5p_2,\ \text{and}\ x_2 = 140 + 0.5p_1 - p_2.$$

Find the prices p_1 and p_2 so that the total revenue from the sales of the two products is maximized.

Solution The revenue function is $R(p_1,p_2) = x_1p_1 + x_2p_2$. So,

$$R(p_1,p_2) = p_1\Big(100 - 2p_1 + 1.5p_2\Big) + p_2\Big(140 + 0.5p_1 - p_2\Big)$$

$$\Rightarrow R(p_1,p_2) = 100p_1 - 2p_1^2 + 2p_1p_2 + 140p_2 - p_2^2.$$

$$\Rightarrow \frac{\partial R}{\partial p_1} = 100 - 4p_1 + 2p_2 = 0,\ \text{and}\ \frac{\partial R}{\partial p_2} = 2p_1 + 140 - 2p_2 = 0.$$

Solving this system of equations gives $p_1 = 120$ and $p_2 = 190$. This is the only one critical number. Now,

$$R_{p_1p_1} = -4,\ \ R_{p_2p_2} = -2,\ \ R_{p_1p_2} = 2 \Rightarrow D(p_1,p_2) = (-4)(-2)-2^2 = 8-4 = 4.$$

We see that $D(120,190) > 0$, but $R_{p_1p_1} < 0$. Thus, the revenue is maximized when $p_1 = \$120$ and $p_2 = \$190$.

Notes:

Example 5.5.9

The demand functions for two products are given by

$$x_1 = 200(p_2 - p_1), \text{ and } x_2 = 500 + 100p_1 - 180p_2.$$

If the costs of producing the two products are \$0.50 and \$0.75 per unit, respectively, find the prices p_1 and p_2 so that the total profit from the sales of the two products is maximized.

Solution The cost and revenue functions are

$$C = 0.5x_1 + 0.75x_2 = 0.5(200)(p_2 - p_1) + 0.75(500 + 100p_1 - 180p_2)$$

$$\Rightarrow C = 375 - 25p_1 - 35p_2.$$

$$R = p_1x_1 + p_2x_2 = p_1(200)(p_2 - p_1) + p_2(500 + 100p_1 - 180p_2)$$

$$\Rightarrow R = -200p_1^2 - 180p_2^2 + 300p_1p_2 + 500p_2.$$

So, the profit function is

$$P = R - C = -200p_1^2 - 180p_2^2 + 300p_1p_2 + 25p_1 + 535p_2 - 375.$$

$$\frac{\partial P}{\partial p_1} = -400p_1 + 300p_2 + 25 = 0, \text{ and } \frac{\partial P}{\partial p_2} = 300p_1 - 360p_2 + 535 = 0.$$

Solving these equations gives $p_1 = \$3.14$ and $p_2 = \$4.10$. Note: We need to test to see if this is a maximum. The maximum profit is $P(3.14, 4.10) = \$761.48$.

Notes:

Exercises 5.5

Problems

In Exercises 1 – 4, linearize the following functions at the indicated point. Notice that the tangent plane to $f(x, y)$ at the point $\left(a, b, f(a, b)\right)$ is given by

$$T(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

1. $f(x, y) = \ln(1 + x + y)$ at $(0, 0)$; $(1, 0)$; $(1, 1)$; and $(1, -1)$.
2. $f(x, y) = (x + e^y)^{1/2}$ at $(3, 0)$.
3. $f(x, y) = x^2 + y^2 + 2x - 2y$ at $(1, 1)$; then approximate $f(0.9, 1.1)$ using the linearization.
4. $P(x, y) = 100x^{1/4}y^{3/4}$ at $(81, 16)$; approximate $P(80, 17)$ using this linearization.

In Exercises 5 – 7, say $(1, 2)$ is a critical point of $z = f(x, y)$ (that is, $f_x(1, 2) = f_y(1, 2) = 0$) and f has continuous second partial derivatives; what can be said about the critical point of f at $(1, 2)$ in each case below?

5. a) $f_{xx}(1, 2) = 1$; $f_{xy}(1, 2) = -2$; $f_{yy}(1, 2) = 1$.
   b) $f_{xx}(1, 2) = -1$; $f_{xy}(1, 2) = 1$; $f_{yy}(1, 2) = -2$.
   c) $f_{xx}(1, 2) = -1$; $f_{xy}(1, 2) = 0$; $f_{yy}(1, 2) = 1$.
6. a) $f_{xx}(1, 2) = -2$; $f_{xy}(1, 2) = -1$; $f_{yy}(1, 2) = -1$.
   b) $f_{xx}(1, 2) = 1$; $f_{xy}(1, 2) = 0$; $f_{yy}(1, 2) = 1$.
   c) $f_{xx}(1, 2) = 2$; $f_{xy}(1, 2) = 2$; $f_{yy}(1, 2) = 1$.

In Exercises 7 – 14, identify the critical points of the function and use the Second Derivative Test for Extrema to classify each as a local minimum, local maximum or saddle point; and if the test fails, (usually if $D = 0$) use direct methods to establish the type of critical point; and where possible, indicate if the local extremum is in fact an absolute extremum.
HINT: all critical points occur at rational or integral values of x and y.

7. $f(x, y) = x^3 + x^2 - y^3 + y^2$.
8. $f(x, y) = xy(4 - 3x - 3y)$.
9. $f(x, y) = \dfrac{1}{x} + xy - \dfrac{8}{y}$.
10. $f(x, y) = e^y(y^2 - x^2)$.
11. $f(x, y) = x^3 - 6xy + y^3$.
12. $f(x, y) = 6x^2y - 3x^2 - 12y^2 + 8y^3$.
13. $f(x, y) = 6xy - x^2 - 6y^3$.
14. $f(x, y) = x^2 - 2axy + y^2$, where a is a real constant. Consider different values of a and classify the critical point(s) of f accordingly by answering the following.
    a) Show that f has one critical point at $(0, 0)$ other than if $|a| = 1$; when $a = 1$, all points along the line $y = x$ are critical and when $a = -1$, all points along the line $y = -x$ are critical.
    b) Find D, the Hessian of f in terms of a, and use it to classify the critical point(s) of f when
    i) $|a| = 1$; ii) $|a| < 1$; iii) $|a| > 1$.
15. Find the point on the plane $3x + 2y + z = 1$ closest to $(1, 1, 1)$. How far apart are the two points?
16. Show that a cube has the smallest surface area of all rectangular solids of a given volume. Include showing that an absolute minimum has been found. [NOTICE: a cube has the largest volume of all rectangular solids with a given surface area.]

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5.6 Lagrange Multipliers

Predicting the behaviour of the consumers is one of the most important, but challenging problems in economics and marketing. The problems involve maximizing or minimizing certain functions under certain constraints on the problems. This type of problems is called constrained optimization problem.

Basic Idea: We start with a problem involving two independent variables. The problem is to find maximum and/or minimum values of a differentiable function f of two variables x and y called the objective function, with restriction that x and y must lie on a constraint curve C in the xy-plane given by $g(x, y) = 0$. See Figure 5.29.

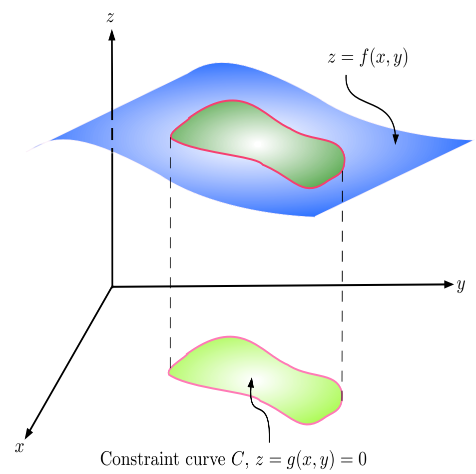

Figure 5.29: Objective Function and Constraint Curve.

The method we use is called Lagrange Multiplier, named after the French mathematician Joseph Louis Lagrange (1736-1813). It is usually used for the problem of the form

Find the extrema for $z = f(x, y)$, subject to $g(x, y) = 0$.

Procedure 5.6.1 (Lagrange Multipliers)**.** *If $f(x, y)$ has a maximum or minimum subject to the constraint $g(x, y) = 0$, then it will occur at one of the critical numbers of the function F defined by*

$$F(x, y, \lambda) = f(x, y) - \lambda g(x, y).$$

The variable λ (the lower case Greek letter lambda) is called a **Lagrange Multiplier**. *To find the maximum or minimum of f, use the following steps.*

1. *Solve the following system of equations*

$$\begin{cases} F_x(x, y, \lambda) = 0 \\ F_y(x, y, \lambda) = 0 \\ F_\lambda(x, y, \lambda) = 0 \end{cases}$$

That is,

$$\begin{cases} f_x(x, y) & = & \lambda g_x(x, y) \\ f_y(x, y) & = & \lambda g_y(x, y) \\ g(x, y) & = & 0 \end{cases}$$

2. *Evaluate f at each solution (critical numbers of F), and end points, if known, obtained in the previous step. The greatest value gives the maximum value of the function f and the smallest value gives the minimum value of the function f, subject to the constraint $g(x, y) = 0$.*

Notes:

Notes:

- The maximum and minimum we get are the global extrema. Sometimes, for most of the problems we are doing in this course, the extrema occur at the critical numbers of F. Most problems have only one critical number.
- λ is the rate of change of the optimal value of f with respect to the level of constraint, that is, with respect to the curve C in $g(x, y) = 0$.
- For function of three variables, $f(x, y, z)$, we add one more line

$$\begin{cases} f_x(x, y, z) &= \lambda g_x(x, y, z) \\ f_y(x, y, z) &= \lambda g_y(x, y, z) \\ f_z(x, y, z) &= \lambda g_z(x, y, z) \\ g(x, y, z) &= 0 \end{cases}$$

- Omit any constrained version of second derivatives test. Also, omit constraint problems involving more than one constraints.

Example 5.6.1
Minimize $f(x, y) = x^2 + y^2$ subject to $x + y = 1$.

Solution First, we write $g(x, y) = x + y - 1 = 0$, which is the constraint that we work with.

$$\begin{cases} f_x(x, y) &= \lambda g_x(x, y) \\ f_y(x, y) &= \lambda g_y(x, y) \\ g(x, y) &= 0 \end{cases} \Rightarrow \begin{cases} 2x &= \lambda \\ 2y &= \lambda \\ x + y &= 1 \end{cases} \Rightarrow \begin{cases} x &= y \\ x &= \frac{1}{2} \\ y &= \frac{1}{2} \end{cases}$$

Then, $f\left(\frac{1}{2}, \frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$. So, the minimum value of f is $\frac{1}{2}$ found at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Example 5.6.2
Find minimum and maximum values of $f(x, y, z) = x - 2y + 2z$ subject to $x^2 + y^2 + z^2 = 1$.

Solution First, we write $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$, which

Notes:

is the constraint that we work with.

$$\begin{cases} f_x(x,y,z) &= \lambda g_x(x,y,z) \\ f_y(x,y,z) &= \lambda g_y(x,y,z) \\ f_z(x,y,z) &= \lambda g_z(x,y,z) \\ g(x,y,z) &= 0 \end{cases} \Rightarrow \begin{cases} 1 &= 2x\lambda \\ -2 &= 2y\lambda \\ 2 &= 2z\lambda \\ x^2+y^2+z^2 &= 1 \end{cases} \Rightarrow \begin{cases} \lambda &= \frac{1}{2x} \\ \lambda &= -\frac{1}{y} \\ \lambda &= \frac{1}{z} \end{cases}$$

The first two equations gives $y = -2x$, and the first and the last equations gives $z = 2x$. Plug these into the constraints gives $x^2 + 4x^2 + 4x^2 = 1 \Rightarrow x = \pm\frac{1}{3}$.

$$x = \frac{1}{3} \Rightarrow y = -\frac{2}{3}, \text{ and } z = \frac{2}{3} \Rightarrow \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right).$$

$$x = -\frac{1}{3} \Rightarrow y = \frac{2}{3}, \text{ and } z = -\frac{2}{3} \Rightarrow \left(-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right).$$

Then, $f\left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right) = \frac{1}{3} - 2\left(-\frac{2}{3}\right) + 2\left(\frac{2}{3}\right) = 3$, and $f\left(-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right) = -\frac{1}{3} - 2\left(\frac{2}{3}\right) + 2\left(-\frac{2}{3}\right) = -3$. So, the minimum value of f is -3 and the maximum value of f is 3.

Example 5.6.3

Let a Cobb-Douglas production function be defined by $P(K, L) = 300K^{2/3}L^{1/3}$, where K is the number of units of the capital and L is the number of units of the labour used in the production per hour. Suppose that the cost of labour is \$20 per unit and the cost of capital is \$10 per unit. Find the amount of labour and capital used to maximize production if a total \$1200 per hour is available for the labour and the capital.

Solution This problem is to maximize $P(K, L)$ subject to $10K + 20L = 1200$. Here, $G(K, L) = 10K + 20L - 1200$.

$$\begin{cases} P_K(K,L) &= \lambda G_K(K,L) \\ P_L(K,L) &= \lambda G_L(K,L) \\ G(K,L) &= 0 \end{cases} \Rightarrow \begin{cases} 200K^{-1/3}L^{1/3} &= 10\lambda \\ 100K^{2/3}L^{-2/3} &= 20\lambda \\ 10K + 20L &= 1200 \end{cases}$$

The first and the second equations gives $400K^{-1/3}L^{1/3} = 100K^{2/3}L^{-2/3} \Rightarrow 4L = K$. Plug into the third equation, we get $L = 20$ and $K = 80$.

Notes:

- We can find $\lambda \approx 12.6$ from one of those equations. It is the marginal productivity of money. It means that if we increase the budget to

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Notes:

\$1300 per hour, then the production would go up by $100 \times 12.6 = 1260$ units.

- P_K is the marginal productivity of labour.
- P_L is the marginal productivity of capital.

Example 5.6.4

A rectangular box in to be built of a wooden base and heavy cardboard with no top. Find the dimensions of the cheapest box that can hold a volume of 12 cubic metres, assuming that wood is three times more expensive than the cardboard.

Solution Volume $V = xyz$ and we want $V = 12$. Cost $C(x, y, z) = 3xy + 2xz + 2yz$. We want to minimize $C(x, y, z)$ subject to $xyz = 12$. Here, $g(x, y, z) = xyz - 12 = 0$

$$\begin{cases} C_x(x,y,z) &= \lambda g_x(x,y,z) \\ C_y(x,y,z) &= \lambda g_y(x,y,z) \\ C_z(x,y,z) &= \lambda g_z(x,y,z) \\ g(x,y,z) &= 0 \end{cases} \Rightarrow \begin{cases} 3y+2z &= yz\lambda \\ 3x+2z &= xz\lambda \\ 2x+2y &= xy\lambda \\ xyz &= 12 \end{cases} \Rightarrow \begin{cases} 3xy+2xz &= xyz\lambda \\ 3xy+2yz &= xyz\lambda \\ 2xz+2yz &= xyz\lambda \\ xyz &= 12 \end{cases}$$

Solving these equations gives $x = y$ and $z = \frac{3}{2}y$. These gives $x = 2,\ y = 2,\ z = 3$. The cheapest box has base 2 m by 2 m and is 3 m tall.

Notes:

- Solving for λ from one of those equations gives $\lambda = \$2/m^3$. A box of 13 m^3 would cost about \$2 more.
- The cost for building the cheapest box is then $C(2, 2, 3) = \$36$.

Example 5.6.5

A computer manufacturing company has an order for 200 computers, and wants to distribute its manufacture between two of its plants, A and B. Let x_a and x_b be the outputs of plant A and B, respectively, and suppose that the total cost function is given by $C(x_a, x_b) = 2x_a^2 + x_a x_b + x_b^2 + 200$. How should the output be distributed in order to minimize the cost?

Solution This problem is to maximize C subject to $x_a + x_b = 200$. Here, $g(x_a, x_b) = x_a + x_b - 200$.

$$\begin{cases} C_{x_a}(x_a,x_b) &= \lambda g_{x_a}(x_a,x_b) \\ C_{x_b}(x_a,x_b) &= \lambda g_{x_b}(x_a,x_b) \\ g(x_a,x_b) &= 0 \end{cases} \Rightarrow \begin{cases} 4x_a + x_b &= \lambda \\ x_a + 2x_b &= \lambda \\ x_a + x_b &= 200 \end{cases}$$

Notes:

From the first two equations, we get $3x_a - x_b = 0 \Rightarrow x_b = 3x_a$. Plug this into the third equation yields $x_a = 50 \Rightarrow x_b = 150$. So, the plant A should manufacture 50 computers, and plant B should make the rest, which is 150 computers.

Example 5.6.6

A small firm in Vancouver wants to produce a given quantity of P_0 of output of its product in the cheapest way. If there are two input factors K and L, and their prices per unit are fixed at p_K and p_L, respectively, discuss the economic significance of combining input to achieve the least cost.

Solution Let $P = f(K, L)$ be the production function. We must minimize the cost function $C(K, L) = Kp_K + Lp_L$ subject to $P_0 = f(K, L)$. Here, $G(K, L) = P_0 - f(K, L)$.

$$\begin{cases} C_K(K,L) &= \lambda G_K(K,L) \\ C_L(K,L) &= \lambda G_L(K,L) \\ G(K,L) &= 0 \end{cases} \Rightarrow \begin{cases} p_K &= \lambda \dfrac{\partial}{\partial K}\Big[f(K,L)\Big] \\ p_L &= \lambda \dfrac{\partial}{\partial L}\Big[f(K,L)\Big] \\ f(K,L) &= P_0 \end{cases}$$

From the first two equations, we get $\lambda = \dfrac{p_K}{\frac{\partial}{\partial K}\Big[f(K,L)\Big]} = \dfrac{p_L}{\frac{\partial}{\partial L}\Big[f(K,L)\Big]} \Rightarrow$

$\dfrac{p_K}{p_L} = \dfrac{\frac{\partial}{\partial K}\Big[f(K,L)\Big]}{\frac{\partial}{\partial L}\Big[f(K,L)\Big]}$. So, to achieve the least cost combination of the factors, the ratio of the marginal productivity of the input factors must be equal to the ratio of their corresponding unit prices.

Notes:

Exercises 5.6

Problems

In Exercises 1 – 4, use the method of Lagrange Multipliers to find the maximum and minimum values of the objective function f on the given constraint. Give the value of the Lagrange multiplier λ for each solution.

1. $f(x, y) = x^2 + y$ such that $x^2 + y^2 = 1$.

2. $f(x, y) = x^2 + y^2$ such that $x^2 + y = 1$.

3. $f(x, y) = (x-1)^2+(y-2)^2-4$ such that $3x+5y = 47$.

4. $f(x, y) = x^2 + y^2$ such that $xy = 1$.

5. An example where the Lagrange method fails to find max and min of a function on a constraint: let $f(x, y) = x + y$ such that $\sqrt{x} + \sqrt{y} = 1$.

   a) Find the extreme value of f using the method of Lagrange. Give the x- and y-coordinates of the point, the value of f there and the associated λ-value.

   b) Confirm that the value of f found in part (a) is not as large as $f(1, 0)$ or $f(0, 1)$. The method of Lagrange has failed to find the maximum value of f on this constraint: why?

   c) What is *happening* at the point $\left(\frac{1}{4}, \frac{1}{4}\right)$? Explain.

6: Differential Equations

The process of formulating an equation or multiple equations to describe a physical phenomenon is called *mathematical modeling.* As a simple example, populations of bacteria are often described as "growing exponentially." Looking in a biology text, we might see $P(t) = P_0 e^{kt}$, where $P(t)$ is the bacteria population at time t, P_0 is the initial population at time $t = 0$, and the constant k describes how quickly the population grows. This equation for exponential growth arises from the assumption that the population of bacteria grows at a rate proportional to its size. Recalling that the derivative gives the rate of change of a function, we can describe the growth assumption precisely using the equation $P' = kP$. This equation is called a *differential equation*, and these equations are the subject of the current chapter.

6.1 Graphical and Numerical Solutions to Differential Equations

In Section 1.1, we were introduced to the idea of a differential equation. Given a function $y = f(x)$, we defined a *differential equation* as an equation involving y, x, and derivatives of y. We explored the simple differential equation $y' = 2x$, and saw that a *solution* to a differential equation is simply a function that satisfies the differential equation.

Introduction and Terminology

Definition 6.1.1 Differential Equation

Given a function $y = f(x)$, a **differential equation** is an equation relating x, y, and derivatives of y.

- The variable x is called the **independent variable**.
- The variable y is called the **dependent variable**.
- The **order** of the differential equation is the order of the highest derivative of y that appears in the equation.

Let us return to the simple differential equation

$$y' = 2x.$$

To find a solution, we must find a function whose derivative is $2x$. In other words, we seek an antiderivative of $2x$. The function

$$y = x^2$$

is an antiderivative of $2x$, and solves the differential equation. So do the functions

$$y = x^2 + 1$$

and

$$y = x^2 - 2346.$$

We call the function

$$y = x^2 + C,$$

with C an arbitrary constant of integration, the *general solution* to the differential equation.

In order to specify the value of the integration constant C, we require additional information. For example, if we know that $y(1) = 3$, it follows that $C = 2$. This additional information is called an *initial condition.*

Definition 6.1.2 Initial Value Problem

A differential equation paired with an initial condition (or initial conditions) is called an **initial value problem.**

The solution to an initial value problem is called a **particular solution**. A particular solution does not include arbitrary constants.

The family of solutions to a differential equation that encompasses all possible solutions is called the **general solution** to the differential equation.

Note: A general solution typically includes one or more arbitrary constants. Different values of the constant(s) specify different members in the family of solutions. The particular solution to an initial value problem is the specific member in the family of solutions that corresponds to the given initial condition(s).

Notes:

Example 6.1.1 A simple first-order differential equation
Solve the differential equation $y' = 2y$.

Solution The solution is a function y such that differentiation yields twice the original function. Unlike our starting example, finding the solution here does not involve computing an antiderivative. Notice that "integrating both sides" would yield the result $y = \int 2y\,dx$, which is not useful. Without knowledge of the function y, we can't compute the indefinite integral. Later sections will explore systematic ways to find analytic solutions to simple differential equations. For now, a bit of thought might let us guess the solution

$$y = e^{2x}.$$

Notice that application of the chain rule yields $y' = 2e^{2x} = 2y$. Another solution is given by

$$y = -3e^{2x}.$$

In fact,

$$y = Ce^{2x},$$

where C is any constant, is the *general solution* to the differential equation because $y' = 2Ce^{2x} = 2y$.

If we are provided with a single initial condition, say $y(0) = 3/2$, we can identify $C = 3/2$ so that

$$y = \frac{3}{2}e^{2x}$$

is the *particular solution* to the initial value problem

$$y' = 2y, \text{ with } y(0) = \frac{3}{2}.$$

Figure 6.1 shows various members of the general solution to the differential equation $y' = 2y$. Each C value yields a different member of the family, and a different function. We emphasize the particular solution corresponding to the initial condition $y(0) = 3/2$.

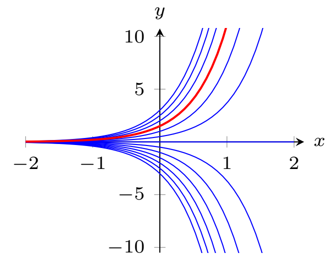

Figure 6.1: General solutions to the differential equation $y' = 2y$, including the particular solution to the initial value problem with $y(0) = 3/2$.

Example 6.1.2 A second-order differential equation
Solve the differential equation $y'' + 9y = 0$.

Solution We seek a function whose second derivative is negative 9 multiplied by the original function. Both $\sin(3x)$ and $\cos(3x)$ have this feature. The general solution to the differential equation is given by

$$y = C_1 \sin(3x) + C_2 \cos(3x),$$

Notes:

where C_1 and C_2 are arbitrary constants. To fully specify a particular solution, we require two additional conditions. For example, the initial conditions $y(0) = 1$ and $y'(0) = 3$ yield $C_1 = C_2 = 1$.

The differential equation in Example 6.1.2 is second order because the equation involves a second derivative. In general, the number of initial conditions required to specify a particular solution depends on the order of the differential equation. For the remainder of the chapter, we restrict our attention to first order differential equations and first order initial value problems.

Example 6.1.3 Verifying a solution to the differential equation
Which of the following is a solution to the differential equation

$$y' + \frac{y}{x} - \sqrt{y} = 0?$$

a) $y = C\left(1 + \ln x\right)^2$ b) $y = \left(\frac{1}{3}x + \frac{C}{\sqrt{x}}\right)^2$ c) $y = Ce^{-3x} + \sqrt{\sin x}$

Solution Verifying a solution to a differential equation is simply an exercise in differentiation and simplification. We substitute each potential solution into the differential equation to see if it satisfies the equation.

a) Testing the potential solution $y = C\left(1 + \ln x\right)^2$:

Differentiating, we have $y' = \dfrac{2C(1 + \ln x)}{x}$. Substituting into the differential equation,

$$\begin{aligned} &\frac{2C(1 + \ln x)}{x} + \frac{C(1 + \ln x)^2}{x} - \sqrt{C}(1 + \ln x) \\ &= (1 + \ln x)\left(\frac{2C}{x} + \frac{C(1 + \ln x)}{x} - \sqrt{C}\right) \\ &\neq 0. \end{aligned}$$

Since it doesn't satisfy the differential equation, $y = C(1 + \ln x)^2$ is *not* a solution.

b) Testing the potential solution $y = \left(\dfrac{1}{3}x + \dfrac{C}{\sqrt{x}}\right)^2$:

Differentiating, we have $y' = 2\left(\dfrac{1}{3}x + \dfrac{C}{\sqrt{x}}\right)\left(\dfrac{1}{3} - \dfrac{C}{2x^{3/2}}\right)$. Substitut-

Notes:

ing into the differential equation,

$$2\left(\frac{1}{3}x+\frac{C}{\sqrt{x}}\right)\left(\frac{1}{3}-\frac{C}{2x^{3/2}}\right)+\frac{1}{x}\left(\frac{1}{3}x+\frac{C}{\sqrt{x}}\right)^2-\left(\frac{1}{3}x+\frac{C}{\sqrt{x}}\right)$$
$$=\left(\frac{1}{3}x+\frac{C}{\sqrt{x}}\right)\left(\frac{2}{3}-\frac{C}{x^{3/2}}+\frac{1}{3}+\frac{C}{x^{3/2}}-1\right)$$
$$=0. \quad \text{(Note how the second parenthetical grouping above reduces to 0.)}$$

Thus $y=\left(\frac{1}{3}x+\frac{C}{\sqrt{x}}\right)^2$ *is* a solution to the differential equation.

c) Testing the potential solution $y=Ce^{-3x}+\sqrt{\sin x}$:

Differentiating, $y'=-3Ce^{-3x}+\dfrac{\cos x}{2\sqrt{\sin x}}$. Substituting into the differential equation,

$$-3Ce^{-3x}+\frac{\cos x}{2\sqrt{\sin x}}+\frac{Ce^{-3x}+\sqrt{\sin x}}{x}-\sqrt{Ce^{-3x}+\sqrt{\sin x}}\neq 0.$$

The function $y=Ce^{-3x}+\sqrt{\sin x}$ is *not* a solution to the differential equation.

Example 6.1.4 Verifying a solution to a differential equation

Verify that $x^2+y^2=Cy$ is a solution to $y'=\dfrac{2xy}{x^2-y^2}$.

Solution The solution in this example is called an *implicit solution.* That means the dependent variable y is a function of x, but has not been explicitly solved for. Verifying the solution still involves differentiation, but we must take the derivatives implicitly. Differentiating, we have

$$2x+2yy'=Cy'.$$

Solving for y', we have

$$y'=\frac{2x}{C-2y}.$$

Notes:

From the solution, we know that $C = \dfrac{x^2+y^2}{y}$. Then

$$\begin{aligned} y' &= \frac{2x}{\dfrac{x^2+y^2}{y} - 2y} \\ &= \frac{2xy}{x^2+y^2-2y^2} \\ &= \frac{2xy}{x^2-y^2}. \end{aligned}$$

We have verified that $x^2 + y^2 = Cy$ is a solution to $y' = \dfrac{2xy}{x^2-y^2}$.

Graphical Solutions to Differential Equations

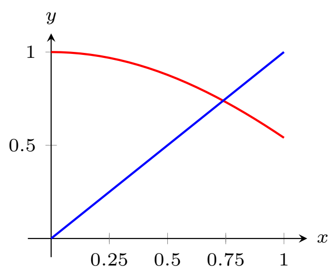

Figure 6.2: Graphically finding an approximate solution to $\cos x = x$.

In the examples we have explored so far, we have found exact forms for the functions that solve the differential equations. Solutions of this type are called *analytic solutions*. Many times a differential equation has a solution, but it is difficult or impossible to find the solution analytically. This is analogous to algebraic equations. The algebraic equation $x^2 + 3x - 1 = 0$ has two real solutions that can be found analytically by using the quadratic formula. The equation $\cos x = x$ has one real solution, but we can't find it analytically. As shown in Figure 6.2, we can find an approximate solution graphically by plotting $\cos x$ and x and observing the x-value of the intersection. We can similarly use graphical tools to understand the qualitative behaviour of solutions to a first order-differential equation.

Consider the first-order differential equation

$$y' = f(x, y).$$

The function f could be any function of the two variables x and y. Written in this way, we can think of the function f as providing a formula to find the slope of a solution at a given point in the xy-plane. In other words, suppose a solution to the differential equation passes through the point (x_0, y_0). At the point (x_0, y_0), the slope of the solution curve will be $f(x_0, y_0)$. Since this calculation of the slope is possible at any point (x, y) where the function $f(x, y)$ is defined, we can produce a plot called a *slope field* (or direction field) that shows the slope of a solution at any point in the xy-plane where the solution is defined. Further, this process can be done purely by working with the differential equation itself. In other

Notes:

words, we can draw a slope field and use it to determine the qualitative behaviour of solutions to a differential equation without having to solve the differential equation.

Definition 6.1.3 Slope Field

A **slope field** for a first-order differential equation $y' = f(x, y)$ is a plot in the xy-plane made up of short line segments or arrows. At each point (x_0, y_0) where $f(x, y)$ is defined, the slope of the line segment is given by $f(x_0, y_0)$. Plots of solutions to a differential equation are tangent to the line segments in the slope field.

Example 6.1.5 Sketching a slope field
Find a slope field for the differential equation $y' = x + y$.

Solution Because the function $f(x, y) = x + y$ is defined for all points (x, y), every point in the xy-plane has an associated line segment. It is not practical to draw an entire slope field by hand, but many tools exist for drawing slope fields on a computer. Here, we explicitly calculate a few of the line segments in the slope field.

- The slope of the line segment at $(0, 0)$ is $f(0, 0) = 0 + 0 = 0$.
- The slope of the line segment at $(1, 1)$ is $f(1, 1) = 1 + 1 = 2$.
- The slope of the line segment at $(1, -1)$ is $f(1, -1) = 1 - 1 = 0$.
- The slope of the line segment at $(-2, 3)$ is $f(-2, -1) = -2 - 1 = -3$.

Though it is possible to continue this process to sketch a slope field, we usually use a computer to make the drawing. Most popular computer algebra systems can draw slope fields. There are also various online tools that can make the drawings. The slope field for $y' = x + y$ is shown in Figure 6.3.

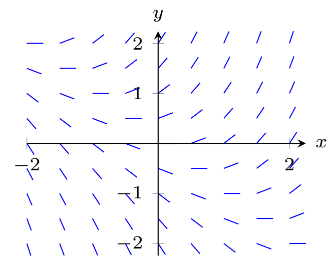

Figure 6.3: Slope field for $y' = x + y$ from Example 6.1.5.

Example 6.1.6 Sketch a graphical solution to an initial value problem
Approximate, with a sketch, the solution to the initial value problem $y' = x + y$, with $y(1) = -1$.

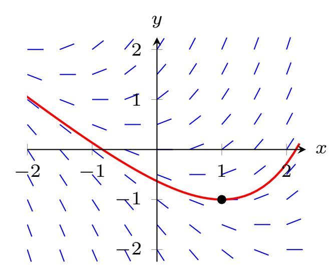

Figure 6.4: Solution to the initial value problem $y' = x+y$, with $y(1) = -1$ from Example 6.1.6

Solution The solution to the initial value problem should be a continuous smooth curve. Using the slope field, we can draw of a sketch of the solution using the following two criteria:

Notes:

1. The solution must pass through the point $(1, -1)$.

2. When the solution passes through a point (x_0, y_0) it must be tangent to the line segment at (x_0, y_0).

Essentially, we sketch a solution to the initial value problem by starting at the point $(1, -1)$ and "following the lines" in either direction. A sketch of the solution is shown in Figure 6.4.

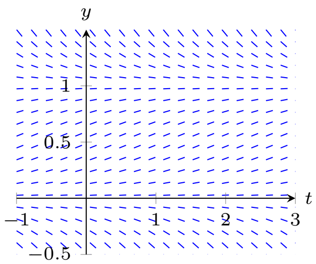

Figure 6.5: Slope field for the logistic differential equation $y' = y(1 - y)$ from Example 6.1.7.

Example 6.1.7 Using a slope field to predict long term behavior
Use the slope field for the differential equation $y' = y(1 - y)$, shown in Figure 6.5, to predict long term behavior of solutions to the equation.

Solution This differential equation, called the *logistic differential equation*, often appears in population biology to describe the size of a population. For that reason, we use t (time) as the independent variable instead of x. We also often restrict attention to non-negative y-values because negative values correspond to a negative population.

Looking at the slope field in Figure 6.5, we can predict long term behaviour for a given initial condition.

- If the initial y-value is negative ($y(0) < 0$), the solution curve must pass though the point $(0, y(0))$ and follow the slope field. We expect the solution y to become more and more negative as time increases. Note that this result is not physically relevant when considering a population.

- If the initial y-value is greater than 0 but less than 1, we expect the solution y to increase and level off at $y = 1$.

- If the initial y-value is greater than 1, we expect the solution y to decrease and level off at $y = 1$.

The slope field for the logistic differential equation, along with representative solution curves, is shown in Figure 6.6. Notice that any solution curve with positive initial value will tend towards the value $y = 1$. We call this the *carrying capacity.*

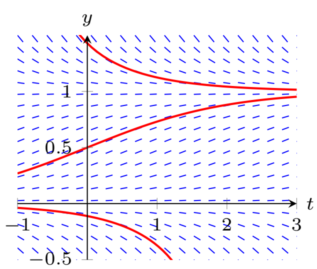

Figure 6.6: Slope field for the logistic differential equation $y' = y(1 - y)$ from Example 6.1.7 with a few representative solution curves.

Numerical Solutions to Differential Equations: Euler's Method

While the slope field is an effective way to understand the qualitative behavior of solutions to a differential equation, it is difficult to use a slope

Notes:

field to make quantitative predictions. For example, if we have the slope field for the differential equation $y' = x+y$ from Example 6.1.5 along with the initial condition $y(0) = 1$, we can understand the qualitative behaviour of the solution to the initial value problem, but will struggle to predict a specific value, $y(2)$ for example, with any degree of confidence. The most straight forward way to predict $y(2)$ is to find the analytic solution to the the initial value problem and evaluate it at $x = 2$. Unfortunately, we have already mentioned that it is impossible to find analytic solutions to many differential equations. In the absence of an analytic solution, a *numerical solution* can serve as an effective tool to make quantitative predictions about the solution to an initial value problem.

There are many techniques for computing numerical solutions to initial value problems. A course in numerical analysis will discuss various techniques along with their strengths and weaknesses. The simplest technique is called *Euler's Method*. Note: Euler's Method is named for Leonhard Euler, a prolific Swiss mathematician during the 1700's. His last name is properly pronounced "oil-er", not "you-ler."

Consider the first-order initial value problem

$$y' = f(x, y), \text{ with } y(x_0) = y_0.$$

Using the definition of the derivative,

$$y'(x) = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}.$$

This notation can be confusing at first, but "$y(x)$" simply means "the y-value of the solution when the x-value is x", and "$y(x+h)$" means "the y-value of the solution when the x-value is $x+h$".

If we remove the limit but restrict h to be "small," we have

$$y'(x) \approx \frac{y(x+h) - y(x)}{h},$$

so that

$$f(x, y) \approx \frac{y(x+h) - y(x)}{h},$$

because $y' = f(x, y)$ according to the differential equation. Rearranging terms,

$$y(x+h) \approx y(x) + h\, f(x, y).$$

This statement says that if we know the solution (y-value) to the initial value problem for some given x-value, we can find an approximation for

Notes:

the solution at the value $x + h$ by taking our y-value and adding h times the function f evaluated at the x and y values. Euler's method uses the initial condition of an initial value problem as the starting point, and then uses the above idea to find approximate values for the solution y at later x-values. The algorithm is summarized in Key Idea 6.1.1.

Key Idea 6.1.1 Euler's Method

Consider the initial value problem

$$y' = f(x, y) \text{ with } y(x_0) = y_0.$$

Let h be a small positive number and N be an integer.

1. For $i = 0, 1, 2, \ldots, N$, define
$$x_i = x_0 + ih.$$
2. The value y_0 is given by the initial condition.
For $i = 0, 1, 2, \ldots, N-1$, define
$$y_{i+1} = y_i + hf(x_i, y_i).$$

This process yields a sequence of $N + 1$ points (x_i, y_i) for $i = 0, 1, 2, \ldots, N$, where (x_i, y_i) is an approximation for $(x_i, y(x_i))$.

Let's practice Euler's Method using a few concrete examples.

Example 6.1.8 Using Euler's Method 1

Find an approximation at $x = 2$ for the solution to $y' = x + y$ with $y(1) = -1$ using Euler's Method with $h = 0.5$.

Solution Our initial condition yields the starting values $x_0 = 1$ and $y_0 = -1$. With $h = 0.5$, it takes $N = 2$ steps to get to $x = 2$. Using steps 1 and 2 from the Euler's Method algorithm,

$x_0 = 1$	$y_0 = -1$
$x_1 = x_0 + h$ $= 1 + 0.5$ $= 1.5$	$y_1 = y_0 + hf(x_0, y_0)$ $= -1 + 0.5(1 - 1)$ $= -1$
$x_2 = x_0 + 2h$ $= 1 + 2(0.5)$ $= 2$	$y_2 = y_1 + hf(x_1, y_1)$ $= -1 + 0.5(1.5 - 1)$ $= -0.75$

Using Euler's method, we find the approximate $y(2) \approx -0.75$.

To help visualize the Euler's method approximation, these three points (connected by line segments) are plotted along with the analytical solution to the initial value problem in Figure 6.7.

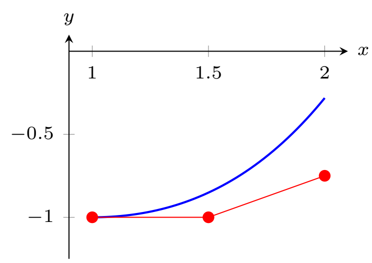

Figure 6.7: Euler's Method approximation to $y' = x + y$ with $y(1) = -1$ from Example 6.1.8, along with the analytical solution to the initial value problem.

This approximation doesn't appear terrific, though it is better than merely guessing. Let's repeat the previous example using a smaller h-value.

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Example 6.1.9 Using Euler's Method 2

Find an approximation on the interval $[1,2]$ for the solution to $y' = x + y$ with $y(1) = -1$ using Euler's Method with $h = 0.25$.

Solution Our initial condition yields the starting values $x_0 = 1$ and $y_0 = -1$. With $h = 0.25$, we need $N = 4$ steps on the interval $[1,2]$ Using steps 1 and 2 from the Euler's Method algorithm (and rounding to 4 decimal points), we have

$x_0 = 1$	$y_0 = -1$
$x_1 = 1.25$	$y_1 = -1 + 0.25(1-1)$ $= -1$
$x_2 = 1.5$	$y_2 = -1 + 0.25(1.25-1)$ $= -0.9375$
$x_3 = 1.75$	$y_3 = -0.9375 + 0.25(1.5 - 0.9375)$ $= -0.7969$
$x_4 = 2$	$y_4 = -0.7969 + 0.25(1.75 - 0.7969)$ $= -0.5586$

Using Euler's method, we find $y(2) \approx -0.5586$.

These five points, along with the points from Example 6.1.8 and the analytic solution, are plotted in Figure 6.8.

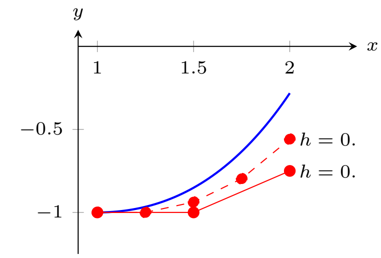

Figure 6.8: Euler's Method approximations to $y' = x + y$ with $y(1) = -1$ from Examples 6.1.8 and 6.1.9, along with the analytical solution.

Using the results from Examples 6.1.8 and 6.1.9, we can make a few observations about Euler's method. First, the Euler approximation generally gets worse as we get farther from the initial condition. This is because Euler's method involves two sources of error. The first comes from the fact that we're using a positive h-value in the derivative approximation instead of using a limit as h approaches zero. Essentially, we're using a linear approximation to the solution y (similar to the process described in Section **??** on Differentials.) This error is often called the *local truncation error.* The second source of error comes from the fact that every step in Euler's method uses the result of the previous step. That means we're using an approximate y-value to approximate the next y-value. Doing this repeatedly causes the errors to build on each other. This second type of error is often called the *propagated* or *accumulated error.*

A second observation is that the Euler approximation is more accurate for smaller h-values. This accuracy comes at a cost, though. Example 6.1.9 is more accurate than Example 6.1.8, but takes twice as many computations. In general, numerical algorithms (even when performed by a computer program) require striking a balance between a desired level of accuracy and the amount of computational effort we are willing to undertake.

Let's do one final example of Euler's Method.

Example 6.1.10 Using Euler's Method 3

Find an approximation for the solution to the logistic differential equation $y' = y(1-y)$ with $y(0) = 0.25$, for $0 \le y \le 4$. Use $N = 10$ steps.

Solution The logistic differential equation is what is called an *autonomous equation.* An autonomous differential equation has no explicit dependence on the independent variable (t in this case). This has no real effect on the application of Euler's method other than the fact that the function $f(t,y)$ is really just a function of y. To take steps in the y variable,

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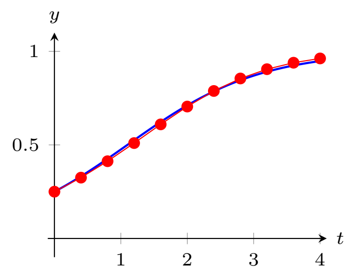


Figure 6.9: Euler's Method approximation to $y' = y(1-y)$ with $y(0) = 0.25$ from Example 6.1.10, along with the analytical solution.

we use

$$y_{i+1} = y_i + hf(t_i, y_i) = y_i + hy_i(1-y_i).$$

Using $N = 10$ steps requires $h = \dfrac{4-0}{10} = 0.4$. Implementing Euler's Method, we have

$x_0 = 0$	$y_0 = 0.25$
$x_1 = 0.4$	$y_1 = 0.25 + 0.4(0.25)(1-0.25)$ $= 0.325$
$x_2 = 0.8$	$y_2 = 0.325 + 0.4(0.325)(1-0.325)$ $= 0.41275$
$x_3 = 1.2$	$y_3 = 0.41275 + 0.4(0.41275)(1-0.41275)$ $= 0.50970$
$x_4 = 1.6$	$y_4 = 0.50970 + 0.4(0.50970)(1-0.50970)$ $= 0.60966$
$x_5 = 2.0$	$y_5 = 0.60966 + 0.4(0.60966)(1-0.60966)$ $= 0.70485$
$x_6 = 2.4$	$y_6 = 0.70485 + 0.4(0.70485)(1-0.70485)$ $= 0.78806$
$x_7 = 2.8$	$y_7 = 0.78806 + 0.4(0.78806)(1-0.78806)$ $= 0.85487$
$x_8 = 3.2$	$y_8 = 0.85487 + 0.4(0.85487)(1-0.85487)$ $= 0.90450$
$x_9 = 3.6$	$y_9 = 0.90450 + 0.4(0.90450)(1-0.90450)$ $= 0.93905$
$x_{10} = 4.0$	$y_{10} = 0.93905 + 0.4(0.93905)(1-0.93905)$ $= 0.96194$

These 11 points, along with the the analytic solution, are plotted in Figure 6.9. Notice how well they seem to match the true solution.

The study of differential equations is a natural extension of the study of derivatives and integrals. The equations themselves involve derivatives, and methods to find analytic solutions often involve finding antiderivatives. In this section, we focus on graphical and numerical techniques to understand solutions to differential equations. We restrict our examples to relatively simple initial value problems that permit analytic solutions to the equations, but we should remember that this is only for comparison purposes. In reality, many differential equations, even some that appear straightforward, do not have solutions we can find analytically. Even so, we can use the techniques presented in this section to understand the behaviour of solutions. In the next two sections, we explore two techniques to find analytic solutions to two different classes of differential equations.

Notes:

Exercises 6.1

Terms and Concepts

1. In your own words, what is an initial value problem, and how is it different than a differential equation?

2. In your own words, describe what it means for a function to be a solution to a differential equation.

3. How can we verify that a function is a solution to a differential equation?

4. Describe the difference between a particular solution and a general solution.

5. Why might we use a graphical or numerical technique to study solutions to a differential equation instead of simply solving the differential equation to find an analytic solution?

6. Describe the considerations that should be made when choosing an h value to use in a numerical method like Euler's Method.

Problems

In Exercises 7 – 10, verify that the given function is a solution to the differential equation or initial value problem.

7. $y = Ce^{-6x^2}$; $\quad y' = -12xy$.

8. $y = x\sin x$; $\quad y' - x\cos x = (x^2+1)\sin x - xy$, with $y(\pi) = 0$.

9. $2x^2 - y^2 = C$; $\quad yy' - 2x = 0$

10. $y = xe^x$; $\quad y'' - 2y' + y = 0$

In Exercises 11 – 12, verify that the given function is a solution to the differential equation and find the C value required to make the function satisfy the initial condition.

11. $y = 4e^{3x}\sin x + Ce^{3x}$; $\quad y' - 3y = 4e^{3x}\cos x$, with $y(0) = 2$

12. $y(x^2 + y) = C$; $\quad 2xy + (x^2 + 2y)y' = 0$, with $y(1) = 2$

In Exercises 13 – 16, sketch a slope field for the given differential equation. Let x and y range between -2 and 2.

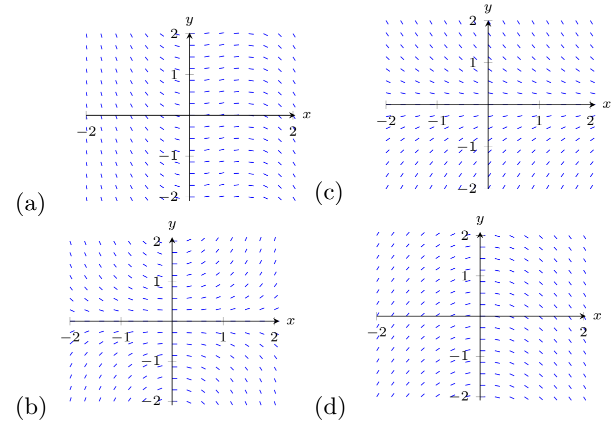
13. $y' = y - x$

14. $y' = \dfrac{x}{2y}$

15. $y' = \sin(\pi y)$

16. $y' = \frac{y}{4}$

In Exercises 17 – 20, match the slope field with the appropriate differential equation.



17. $y' = xy$

18. $y' = -y$

19. $y' = -x$

20. $y' = x(1 - x)$

In Exercises 21 – 24, sketch the slope field for the differential equation, and use it to draw a sketch of the solution to the initial value problem.

21. $y' = \dfrac{y}{x} - y$, with $y(0.5) = 1$.

22. $y' = y\sin x$, with $y(0) = 1$.

23. $y' = y^2 - 3y + 2$, with $y(0) = 2$.

24. $y' = -\dfrac{xy}{1+x^2}$, with $y(0) = 1$.

In Exercises 25 – 28, use Euler's Method to make a table of values that approximates the solution to the initial value problem on the given interval. Use the specified h or N value.

25. $y' = x + 2y$
    $y(0) = 1$
    interval: $[0, 1]$
    $h = 0.25$

26. $y' = xe^{-y}$
    $y(0) = 1$
    interval: $[0, 0.5]$
    $N = 5$

27. $y' = y + \sin x$
    $y(0) = 2$
    interval: $[0, 1]$
    $h = 0.2$

28. $y' = e^{x-y}$
    $y(0) = 0$
    interval: $[0, 2]$
    $h = 0.5$

In Exercises 29 – 30, use the provided solution $y(x)$ and Euler's Method with the $h = 0.2$ and $h = 0.1$ to complete the following table.

x	0.0	0.2	0.4	0.6	0.8	1.0
$y(x)$						
$h = 0.2$						
$h = 0.1$						

29. $y' = xy^2$
    $y(0) = 1$
    solution: $y(x) = \dfrac{2}{1 - x^2}$

30. $y' = xe^{x^2} + \dfrac{1}{2}xy$
    $y(0) = \dfrac{1}{2}$
    solution: $y(x) = \dfrac{1}{2}(x^2 + 1)e^{x^2}$

6.2 Separable Differential Equations

There are specific techniques that can be used to solve specific types of differential equations. This is similar to solving algebraic equations. In algebra, we can use the quadratic formula to solve a quadratic equation, but not a linear or cubic equation. In the same way, techniques that can be used for a specific type of differential equation are often ineffective for a differential equation of a different type. In this section, we describe and practice a technique to solve a class of differential equations called *separable equations.*

Definition 6.2.1 Separable Differential Equation

A **separable differential equation** is one that can be written in the form

$$F(y)\frac{dy}{dx} = G(x),$$

where n is a function that depends only on the dependent variable y, and m is a function that depends only on the independent variable x.

Below, we show a few examples of separable differential equations, along with similar looking equations that are not separable.

Separable	**Not Separable**
1. $\frac{dy}{dx} = x^2 y$	1. $\frac{dy}{dx} = x^2 + y$
2. $y\sqrt{y^2-5}\frac{dy}{dx} - \sin x \cos x = 0$	2. $y\sqrt{y^2-1}\frac{dy}{dx} - \sin x \cos y = 0$
3. $\frac{dy}{dx} = \frac{(x^2+1)e^y}{y}$	3. $\frac{dy}{dx} = \frac{(xy+1)e^y}{y}$

Notice that a separable equation requires that the functions of the dependent and independent variables be multiplied, not added (like example 1 of the not separable column). An alternate definition of a separable differential equation states that an equation is separable if it can be written in the form

$$\frac{dy}{dx} = f(x)g(y),$$

for some functions f and g.

Notes:

Separation of Variables

Let's find a formal solution to the separable equation

$$F(y)\frac{dy}{dx} = G(x).$$

Since the functions on the left and right hand sides of the equation are equal, their antiderivatives should be equal up to an arbitrary constant of integration. That is

$$\int F(y)\frac{dy}{dx} \cdot dx = \int G(x)\,dx$$

Though the integral on the left may look a bit strange, recall that y itself is a function of x. The differential is $dy = \frac{dy}{dx} \cdot dx$. Thus, the above equation becomes

$$\int F(u)\,dy = \int G(x)\,dx$$

This relationship between y and x is an implicit form of the solution to the differential equation. Sometimes (but not always) it is possible to solve for y to find an explicit version of the solution.

Though the technique outlined above is formally correct, what we did essentially amounts to integrating the function n with respect to its variable and integrating the function m with respect to its variable. The informal way to solve a separable equation is to write it in differential forms like follow

$$F(y)\,dy = G(x)\,dx.$$

To solve, we integrate the left hand side with respect to y and the right hand side with respect to x and add a constant of integration. As long as we are able to find the antiderivatives, we can find an implicit form for the solution. Sometimes we are able to solve for y in the implicit solution to find an explicit form of the solution to the differential equation. We practice the technique by solving the three differential equations listed in the separable column above, and conclude by revisiting and finding the general solution to the logistic differential equation from Section 6.1

Example 6.2.1 Solving a Separable Differential Equation
Find the general solution to the differential equation $y' = x^2y$.

Solution Using the informal solution method outlined above, we treat $\frac{dy}{dx}$ as a fraction, and write the separated form of the differential

Notes:

equation as

$$\frac{1}{y}\,dy = x^2\,dx.$$

Integrating the left hand side of the equation with respect to y and the right hand side of the equation with respect to x yields

$$\ln|y| = \frac{1}{3}x^3 + C.$$

This is an implicit form of the solution to the differential equation. Solving for y yields an explicit form for the solution. Exponentiating both sides, we have

$$|y| = e^{x^3/3+C} = e^{x^3/3}e^C.$$

This solution is a bit problematic. First, the absolute value makes the solution difficult to understand. The second issue comes from our desire to find the *general solution.* Recall that a general solution includes all possible solutions to the differential equation. In other words, for any given initial condition, the general solution must include the solution to that specific initial value problem. We can often satisfy any given initial condition by choosing an appropriate C value. When solving separable equations, though, it is possible to lose solutions that have the form $y =$ constant. Notice that $y = 0$ solves the differential equation, but it is not possible to choose a finite C to make our solution look like $y = 0$. Our solution cannot solve the initial value problem $\frac{dy}{dx} = x^2y$, with $y(a) = 0$ (where a is any value). Thus, we haven't actually found a general solution to the problem. We can clean up the solution and recover the missing solution with a bit of clever thought.

Recall the formal definition of the absolute value: $|y| = y$ if $y \geq 0$ and $|y| = -y$ if $y < 0$. Our solution is either $y = e^Ce^{\frac{x^3}{3}}$ or $y = -e^Ce^{\frac{x^3}{3}}$. Further, note that C is constant, so e^C is also constant. If we write our solution as $y = Ae^{\frac{x^3}{3}}$, and allow the constant A to take on either positive or negative values, we incorporate both cases of the absolute value. Finally, if we allow A to be zero, we recover the missing solution discussed above. The best way to express the general solution to our differential equation is

$$y = Ae^{\frac{x^3}{3}}.$$

Note: The indefinite integrals $\int \frac{1}{y}\,dy$ and $\int x^2\,dx$ both produce arbitrary constants. Since both constants are arbitrary, we combine them into

---

Notes:

a single constant of integration.

Note: Missing constant solutions can't always be recovered by cleverly redefining the arbitrary constant. The differential equation $y' = y^2 - 1$ is an example of this fact. Both $y = 1$ and $y = -1$ are constant solutions to this differential equation. Separation of variables yields a solution where $y = 1$ can be attained by choosing an appropriate C value, but $y = -1$ can't. The general solution is the set containing the solution produced by separation of variables *and* the missing solution $y = -1$. We should always be careful to look for missing constant solutions when seeking the general solution to a separable differential equation.

Example 6.2.2 Solving a Separable Initial Value Problem

Solve the initial value problem $(y\sqrt{y^2-5})y' - \sin x \cos x = 0$, with $y(0) = -3$.

Solution We first put the differential equation in separated form

$$y\sqrt{y^2-5}\,dy = \sin x \cos x\,dx.$$

The indefinite integral $\displaystyle\int y\sqrt{y^2-5}\,dy$ requires the substitution $u = y^2 - 5$. Using this substitute yields the antiderivative $\dfrac{1}{3}(y^2-5)^{3/2}$. The indefinite integral $\displaystyle\int \sin x \cos x\,dx$ requires the substitution $u = \sin x$. Using this substitution yields the antiderivative $\dfrac{1}{2}\sin^2 x$. Thus, we have an implicit form of the solution to the differential equation given by

$$\frac{1}{3}(y^2-5)^{3/2} = \frac{1}{2}\sin^2 x + C.$$

The initial condition says that y should be -3 when x is 0, or

$$\frac{1}{3}((-3)^2-5)^{3/2} = \frac{1}{2}\sin^2 0 + C.$$

Evaluating the line above, we find $C = 8/3$, yielding the particular solution to the initial value problem

$$\frac{1}{3}(y^2-5)^{3/2} = \frac{1}{2}\sin^2 x + \frac{8}{3}.$$

Example 6.2.3 Solving a Separable Differential Equation

Find the general solution to the differential equation $\dfrac{dy}{dx} = \dfrac{(x^2+1)e^y}{y}$.

Notes:

Solution We start by observing that there are no constant solutions to this differential equation because there are no constant y values that make the right hand side of the equation identically zero. Thus, we need not worry about losing solutions during the separation of variables process. The separated form of the equation is given by

$$ye^{-y}\,dy = (x^2+1)\,dx.$$

The antiderivative of the left hand side requires Integration by Parts. Evaluating both indefinite integrals yields the implicit solution

$$-(y+1)e^{-y} = \frac{1}{3}x^3 + x + C.$$

Since we cannot solve for y, we cannot find an explicit form of the solution.

Example 6.2.4 Solving the Logistic Differential Equation

Solve the logistic differential equation $\dfrac{dy}{dt} = ky\left(1 - \dfrac{y}{M}\right)$

Solution We looked at a slope field for this equation in Section 6.1 in the specific case of $k = M = 1$. Here, we use separation of variables to find an analytic solution to the more general equation. Notice that the independent variable t does not explicitly appear in the differential equation. We mentioned that an equation of this type is called *autonomous*. All autonomous first order differential equations are separable.

We start by making the observation that both $y = 0$ and $y = M$ are constant solutions to the differential equation. We must check that these solutions are not lost during the separation of variables process. The separated form of the equation is

$$\frac{1}{y\left(1-\dfrac{y}{M}\right)}\,dy = k\,dt.$$

The antiderivative of the left hand side of the equation can be found by making use of partial fractions. Using the techniques discussed in Section 2.5, we write

$$\frac{1}{y\left(1-\frac{y}{M}\right)} = \frac{1}{y} + \frac{1}{M-y}.$$

Then an implicit form of the solution is given by

$$\ln|y| - \ln|M-y| = kt + C.$$

Notes:

Combining the logarithms,

$$\ln\left|\frac{y}{M-y}\right| = kt + C.$$

Similarly to Example 6.2.1, we can write

$$\frac{y}{M-y} = Ae^{kt}.$$

Letting A take on positive values or negative values incorporates both cases of the absolute value. This is another implicit form of the solution. Solving for y gives the explicit form

$$y = \frac{M}{1+be^{-kt}},$$

where b is an arbitrary constant. Notice that $b = 0$ recovers the constant solution $y = M$. The constant solution $y = 0$ cannot be produced with a finite b value, and has been lost. The general solution the logistic differential equation is the set containing $y = \frac{M}{1+be^{-kt}}$ and $y = 0$.

Note: Solving for y initially yields the explicit solution $y = \frac{AMe^{kt}}{1+Ae^{kt}}$. Dividing numerator and denominator by Ae^{kt} and defining $b = 1/A$ yields the commonly presented form of the solution given in Example 6.2.4.

Notes:

Exercises 6.2

Problems

In Exercises 1 – 4, decide whether the differential equation is separable or not separable. If the equation is separable. write it in separated form.

1. $y' = y^2 - y$

2. $xy' + x^2 y = \dfrac{\sin x}{x - y}$

3. $(y+3)y' + (\ln x)y' - x\sin y = (y+3)\ln x$

4. $y' - x^2\cos y + y = \cos y - x^2 y$

In Exercises 5 – 12, find the general solution to the separable differential equation. Be sure to check for missing constant solutions.

5. $y' + 1 - y^2 = 0$

6. $y' = y - 2$

7. $xy' = 4y$

8. $yy' = 4x$

9. $e^x yy' = e^{-y} + e^{-2x-y}$

10. $(x^2+1)y' = \dfrac{x}{y-1}$

11. $y' = \dfrac{x\sqrt{1-4y^2}}{x^4 + 2x^2 + 2}$

12. $(e^x + e^{-x})y' = y^2$

In Exercises 13 – 20, find the particular solution to the separable initial value problem.

13. $y' = \dfrac{\sin x}{\cos y}$, with $y(0) = \dfrac{\pi}{2}$

14. $y' = \dfrac{x^2}{1 - y^2}$, with $y(0) = 1$

15. $y' = \dfrac{2x}{y + x^2 y}$, with $y(0) = -4$

16. $x + ye^{-x}y' = 0$, with $y(0) = -2$

17. $y' = \dfrac{x\ln(x^2+1)}{y-1}$, with $y(0) = 2$

18. $\sqrt{1-x^2}\,y' - \dfrac{\arcsin x}{y\cos(y^2)} = 0$, with $y(0) = \sqrt{\dfrac{7\pi}{6}}$

19. $y' = (\cos^2 x)(\cos^2 2y)$, with $y(0) = 0$

20. $y' = \dfrac{y^2\sqrt{1-y^2}}{x}$, with $y(0) = 1$

6.3 First Order Linear Differential Equations

In the previous section, we explored a specific technique to solve a specific type of differential equation called a separable differential equation. In this section, we develop and practice a technique to solve a type of differential equation called a *first order linear* differential equation.

Recall than a linear algebraic equation in one variable is one that can be written $ax + b = 0$, where a and b are real numbers. Notice that the variable x appears to the first power. The equations $\sqrt{x} + 1 = 0$ and $\sin(x) - 3x = 0$ are both nonlinear. A linear differential equation is one in which the dependent variable and its derivatives appear only to the first power. We focus on first order equations, which involve first (but not higher order) derivatives of the dependent variable.

Definition 6.3.1 First Order Linear Differential Equation

A **first order linear differential equation** is a differential equation that can be written in the form

$$\frac{dy}{dx} + p(x)y = q(x),$$

where p and q are arbitrary functions of the independent variable x.

Example 6.3.1 Classifying Differential Equations

Classify each differential equation as first order linear, separable, both, or neither.

(a) $y' = xy$

(b) $y' = e^y + 3x$

(c) $y' - (\cos x)y = \cos x$

(d) $yy' - 3xy = 4\ln x$

Solution (a) Both. We identify $p(x) = -x$ and $q(x) = 0$. The separated form of the equation is $\dfrac{dy}{y} = x\,dx$.

(b) Neither. The e^y term makes the equation nonlinear. Because of the addition, it is not possible to write the equation in separated form.

(c) First order linear. We identify $p(x) = -\cos x$ and $q(x) = \cos x$. The equation cannot be written in separated form.

Notes:

(d) Neither. Notice that dividing by y results in the nonlinear term $\dfrac{4\ln x}{y}$. It is not possible to write the equation in separated form.

Notice that linearity depends on the dependent variable y, not the independent variable x. The functions $p(x)$ and $q(x)$ need not be linear, as demonstrated in part (c) of Example 6.3.1. Neither $\cos x$ nor $\sin x$ are linear functions of x, but the differential equation is still linear.

Solving First Order Linear Equations

Before working out a general technique for solving first order linear differential equations, we look at a specific example. Consider the differential equation

$$\frac{d}{dx}(xy) = \sin x \cos x.$$

This is an easy differential equation to solve. On the left, the antiderivative of the derivative is simply the function xy. Using the substitution $u = \sin x$ on the right and integrating results in the implicit solution

$$xy = \frac{1}{2}\sin^2 x + C.$$

Solving for y yields the explicit solution

$$y = \frac{\sin^2 x}{2x} + \frac{C}{x}.$$

Though not obvious, the differential equation above is actually a linear differential equation. Using the product rule and implicit differentiation, we can write $\dfrac{d}{dx}(xy) = x\dfrac{dy}{dx} + y$. Our original differential equation can be written

$$x\frac{dy}{dx} + y = \sin x \cos x.$$

If we divide by x, we have

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{\sin x \cos x}{x},$$

which matches the form in Definition 6.3.1. Reversing our steps would lead us back to the original form our our differential equation.

Note: In the examples in the previous section, we performed operations on the arbitrary constant C, but still called the result C. The justification is that the result after the operation is *still* an arbitrary constant. Here,

Notes:

we divide C by x, so the result depends explicitly on the independent variable x. Since C/x is *not* constant, we can't just call it C.

Consider the first order linear equation

$$\frac{dy}{dx} + p(x)y = q(x).$$

Let's call the integrating factor $\mu(x)$. We multiply both sides of the differential equation by $\mu(x)$ to get

$$\mu(x)\left(\frac{dy}{dx} + p(x)y\right) = \mu(x)q(x).$$

Our goal is to choose $\mu(x)$ so that the left hand side of the differential equation looks like the result of a Product Rule. The left hand side of the equation is

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y.$$

Using the Product Rule and Implicit Differentiation,

$$\frac{d}{dx}\big(\mu(x)y\big) = \frac{d\mu}{dx}y + \mu(x)\frac{dy}{dx}.$$

Note: Though we use $\mu(x)$ for our integrating factor, the symbol is unimportant. The notation $\mu(x)$ is a common choice, but other texts my use $\alpha(x), I(x)$, or some other symbol to designate the integrating factor. Equating $\frac{d}{dx}\big(\mu(x)y\big)$ and $\mu(x)\left(\frac{dy}{dx} + p(x)y\right)$ gives

$$\frac{d\mu}{dx}y + \mu(x)\frac{dy}{dx} = \mu(x)\frac{dy}{dx} + \mu(x)p(x)y,$$

which is equivalent to

$$\frac{d\mu}{dx} = \mu(x)p(x).$$

Note: Following the steps outlined in the previous section, we should technically end up with $\mu(x) = Ce^{\int p(x)\,dx}$, where C is an arbitrary constant. Because we multiply both sides of the differential equation by $\mu(x)$, the arbitrary constant cancels, and we omit it when finding the integrating factor. In order for the integrating factor $\mu(x)$ to perform its job, it must solve the differential equation above. But that differential equation is separable, so we can solve it. The separated form is

$$\frac{d\mu}{\mu} = p(x)\,dx.$$

Notes:

Integrating,

$$\ln \mu = \int p(x)\,dx,$$

or

$$\mu(x) = e^{\int p(x)\,dx}.$$

If $\mu(x)$ is chosen this way, after multiplying by $\mu(x)$, we can always write the differential equation in the form

$$\frac{d}{dx}\big(\mu(x)y\big) = \mu(x)q(x).$$

Integrating and solving for y, the explicit solution is

$$y = \frac{1}{\mu(x)}\int \big(\mu(x)q(x)\big)\,dx.$$

Though this formula can be used to write down the solution to a first order linear equation, we shy away from simply memorizing a formula. The process is lost, and it's easy to forget the formula. Rather, we always always follow the steps outlined in Key Idea 6.3.1 when solving equations of this type.

Key Idea 6.3.1 Solving First Order Linear Equations

1. Write the differential equation in the form

$$\frac{dy}{dx} + p(x)y = q(x).$$

2. Compute the integrating factor

$$\mu(x) = e^{\int p(x)\,dx}.$$

3. Multiply both sides of the differential equation by $\mu(x)$, and condense the left hand side to get

$$\frac{d}{dx}\big(\mu(x)y\big) = \mu(x)q(x).$$

4. Integrate both sides of the differential equation with respect to x, taking care to remember the arbitrary constant.
5. Solve for y to find the explicit solution to the differential equation.

Notes:

Let's practice the process by solving the two first order linear differential equations from Example 6.3.1.

Example 6.3.2 Solving a First Order Linear Equation
Find the general solution to $y' = xy$.

Solution We solve by following the steps in Key Idea 6.3.1. Unlike the process for solving separable equations, we need not worry about losing constant solutions. The answer we find *will* be the general solution to the differential equation. We first write the equation in the form

$$\frac{dy}{dx} - xy = 0.$$

By identifying $p(x) = -x$, we can compute the integrating factor

$$\mu(x) = e^{\int -x\,dx} = e^{-\frac{1}{2}x^2}.$$

Multiplying both side of the differential equation by $\mu(x)$, we have

$$e^{-\frac{1}{2}x^2}\left(\frac{dy}{dx} - xy\right) = 0.$$

The left hand side of the differential equation condenses to yield

$$\frac{d}{dx}\left(e^{-\frac{1}{2}x^2}y\right) = 0.$$

We integrate both sides with respect to x to find the implicit solution

$$e^{-\frac{1}{2}x^2}y = C,$$

or the explicit solution

$$y = Ce^{\frac{1}{2}x^2}.$$

Note: The step where the left hand side of the differential equation condenses to the derivative of a product can feel a bit magical. The reality is that we choose $\mu(x)$ so that we can get exactly this condensing behaviour. It's not magic, it's math! If you're still skeptical, try using the Product Rule and Implicit Differentiation to evaluate $\frac{d}{dx}\left(e^{-\frac{1}{2}x^2}y\right)$, and verify that it becomes $e^{-\frac{1}{2}x^2}\left(\frac{dy}{dx} - xy\right)$.

Notes:

Example 6.3.3 Solving a First Order Linear Equation
Find the general solution to $y' - (\cos x)y = \cos x$.

Solution The differential equation is already in the correct form. The integrating factor is given by

$$\mu(x) = e^{-\int \cos x\, dx} = e^{-\sin x}.$$

Multiplying both sides of the equation by the integrating factor and condensing,

$$\frac{d}{dx}\left(e^{-\sin x}y\right) = (\cos x)e^{-\sin x}$$

Using the substitution $u = -\sin x$, we can integrate to find the implicit solution

$$e^{-\sin x}y = -e^{-\sin x} + C.$$

The explicit form of the general solution is

$$y = -1 + Ce^{\sin x}.$$

Notes:

We continue our practice by finding the particular solution to an initial value problem.

Example 6.3.4 Solving a First Order Linear Initial Value Problem

Solve the initial value problem $xy' - y = x^3 \ln x$, with $y(1) = 0$.

Solution We first divide by x to get

$$\frac{dy}{dx} - \frac{1}{x}y = x^2 \ln x.$$

The integrating factor is given by

$$\begin{aligned} \mu(x) &= e^{\int -\frac{1}{x}\,dx} \\ &= e^{-\ln x} \\ &= e^{\ln x^{-1}} \\ &= x^{-1}. \end{aligned}$$

Multiplying both sides of the differential equation by the integrating factor and condensing the left hand side, we have

$$\frac{d}{dx}\left(\frac{y}{x}\right) = x \ln x.$$

Using Integrating by Parts to find the antiderivative of $x \ln x$, we find the implicit solution

$$\frac{y}{x} = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C.$$

Solving for y, the explicit solution is

$$y = \frac{1}{2}x^3 \ln x - \frac{1}{4}x^3 + Cx.$$

The initial condition $y(1) = 0$ yields $C = 1/4$. The solution to the initial value problem is

$$y = \frac{1}{2}x^3 \ln x - \frac{1}{4}x^3 + \frac{1}{4}x.$$

Differential equations are a valuable tool for exploring various physical problems. This process of using equations to describe real world situations is called mathematical modeling, and is the topic of the next section. The

Notes:

last two examples in this section begin our discussion of mathematical modeling.

Example 6.3.5 A Falling Object Without Air Resistance
Suppose an object with mass m is dropped from an airplane. Find and solve a differential equation describing the vertical velocity of the object assuming no air resistance.

Solution The basic physical law at play is Newton's second law,

$$\text{mass} \times \text{acceleration} = \text{the sum of the forces}.$$

Using the fact that acceleration is the derivative of velocity, mass $\times$ acceleration can be writing mv'. In the absence of air resistance, the only force of interest is the force due to gravity. This force is approximately constant, and is given by mg, where g is the gravitational constant. The word equation above can be written as the differential equation

$$m\frac{dv}{dt} = mg.$$

Because g is constant, this differential equation is simply an integration problem, and we find

$$v = gt + C.$$

Since $v = C$ with $t = 0$, we see that the arbitrary constant here corresponds to the initial vertical velocity of the object.

The process of mathematical modeling does not stop simply because we have found an answer. We must examine the answer to see how well it can describe real world observations. In the previous example, the answer may be somewhat useful for short times, but intuition tells us that something is missing. Our answer says that a falling object's velocity will increase linearly as a function of time, but we know that a falling object does not speed up indefinitely. In order to more fully describe real world behaviour, our mathematical model must be revised.

Example 6.3.6 A Falling Object with Air Resistance
Suppose an object with mass m is dropped from an airplane. Find and solve a differential equation describing the vertical velocity of the object, taking air resistance into account.

Solution We still begin with Newton's second law, but now we assume that the forces in the object come both from gravity and from air

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resistance. The gravitational force is still given by mg. For air resistance, we assume the force is related to the velocity of the object. A simple way to describe this assumption might be kv^p, where k is a proportionality constant and p is a positive real number. The value k depends on various factors such as the density of the object, surface area of the object, and density of the air. The value p affects how changes in the velocity affect the force. Taken together, a function of the form kv^p is often called a *power law*. The differential equation for the velocity is given by

$$m\frac{dv}{dt} = mg - kv^p.$$

(Notice that the force from air resistance opposes motion, and points in the opposite direction as the force from gravity.) This differential equation is separable, and can be written in the separated form

$$\frac{m}{mg - kv^p}\,dv = dt.$$

For arbitrary positive p, the integration is difficult, making this problem hard to solve analytically. In the case that $p = 1$, the differential equation becomes linear, and is easy to solve either using either separation of variables or integrating factor techniques. We assume $p = 1$, and proceed with an integrating factor so we can continue practicing the process. Writing

$$\frac{dv}{dt} + \frac{k}{m}v = g,$$

we identify the integrating factor

$$\mu(t) = e^{\int \frac{k}{m}\,dt} = e^{\frac{k}{m}t}.$$

Then

$$\frac{d}{dt}\left(e^{\frac{k}{m}t}v\right) = ge^{\frac{k}{m}t},$$

so

$$e^{\frac{k}{m}t}v = \frac{mg}{k}e^{\frac{k}{m}t} + C,$$

or

$$v = \frac{mg}{k} + Ce^{-\frac{k}{m}t}.$$

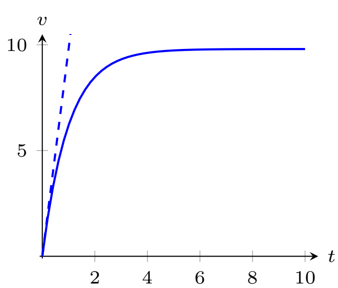

Figure 6.10: The velocity functions from Examples 6.3.5 (dashed) and 6.3.6 (solid) under the assumption that $v(0) = 0$, with $g = 9.8, m = 1$, and $k = 1$.

In the solution above, the exponential term decays as time increases, causing the velocity to approach the constant value mg/k in the limit as t approaches infinity. This value is called the *terminal velocity*. If we

Notes:

assume a zero initial velocity (the object is dropped, not thrown from the plane), the velocities from Examples 6.3.5 and 6.3.6 are given by $v = gt$ and $v = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right)$, respectively. These two functions are shown in Figure 6.10, with $g = 9.8, m = 1$, and $k = 1$. Notice that the two curves agree well for short times, but have dramatically different behaviours as t increases. Part of the art in mathematical modeling is deciding on the level of detail required to answer the question of interest. If we are only interested in the initial behaviour of the falling object, the simple model in Example 6.3.5 may be sufficient. If we are interested in the longer term behaviour of the object, the simple model is not sufficient, and we should consider a more complicated model.

Notes:

Exercises 6.3

Problems

In Exercises 1 – 8, Find the general solution to the first order linear differential equation.

1. $y' = 2y - 3$
2. $x^2 y' + xy = 1$
3. $x^2 y' - xy = 1$
4. $xy' + 4y = x^3 - x$
5. $(\cos^2 x \sin x) y' + (\cos^3 x) y = 1$
6. $\dfrac{y'}{x} = 1 - 2y$
7. $x^3 y' - 3x^3 y = x^4 e^{2x}$
8. $y' + y = 5\sin(2x)$

In Exercises 9 – 16, Find the particular solution to the initial value problem.

9. $y' = y + 2xe^x, \qquad y(0) = 2$
10. $xy' + 2y = x^2 - x + 1, \qquad y(1) = 1$
11. $xy' + (x+2)y = x, \qquad y(1) = 0$
12. $y' + 2y = 0, \qquad y(0) = 3$
13. $(x+1)y' + (x+2)y = 2xe^{-x}, \qquad y(0) = 1$
14. $(\cos x)y' + (\sin x)y = 1, \qquad y(0) = -3$
15. $(x^2 - 1)y' + 2y = (x+1)^2, \qquad y(0) = 2$
16. $xy' - 2y = \dfrac{x^3}{1+x^2}, \qquad y(1) = 0$

In Exercises 17 – 20, classify the differential equation as separable, first order linear, or both, and solve the initial value problem using an appropriate method.

17. $y' = y + yx^2, \qquad y(0) = -5$
18. $xe^y y' = x^2 \sin x, \qquad y(0) = 0$
19. $(x-1)y' + y = x^2 - 1, \qquad y(0) = 2$
20. $y' = y^2 + y - 2, \qquad y(0) = 1$

In Exercises 21 – 22, draw a slope field for the differential equation. Use the slope field to predict the behavior of the solution to the initial value problem for large x values. Solve the initial value problem, and verify your prediction.

21. $y' = x - y, \qquad y(0) = 0$
22. $(X+1)y' + y = \dfrac{1}{x+1}, \qquad y(0) = 2$

6.4 Modeling with Differential Equations

In the first three sections of this chapter, we focused on the basic ideas behind differential equations and the mechanics of solving certain types of differential equations. We have only hinted at their practical use. In this section, we use differential equations for mathematical modeling, the process of using equations to describe real world processes. We explore a few different mathematical models with the goal of gaining an introduction to this large field of applied mathematics.

Models Involving Proportional Change

Some of the simplest differential equation models involve one quantity that changes at a rate proportional to another quantity. In the introduction to this chapter, we considered a population that grows at a rate proportional to the current population. The words in this assumption can be directly translated into a differential equation as shown below.

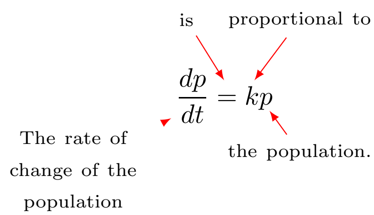

Figure 6.11: Translating words into a differential equation.

There are some key ideas that can be helpful when translating words into a differential equation. Any time we see something about rates or changes, we should think about derivatives. The word "is" usually corresponds to an equal sign in the equation. The words "proportional to" mean we have a constant multiplied by something.

The differential equation in Figure 6.11 is easily solved using separation of variables. We find

$$p = Ce^{kt}.$$

Notice that we need values for both C and k before we can use this formula to predict population size. We require information about the population at two different times in order to fully determine the population model.

Example 6.4.1 Bacterial Growth

Suppose a population of *e-coli* bacteria grows at a rate proportional to the current population. If an initial popluation of 200 bacteria has grown

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to 1600 three hours later, find a function for the size of the population at time t, and use it to predict when the population size will reach 10,000.

Solution We already know that the population at time t is given by $p = Ce^{kt}$ for some C and k. The information about the initial size of the population means that $p(0) = 200$. Thus $C = 200$. Our knowledge of the population size after three hours allows us to solve for k via the equation

$$1600 = 200e^{3k}.$$

Solving this exponential equation yields $k = \ln(8)/3 \approx 0.6931$. The population at time t is given by

$$p = 200e^{(\ln(8)/3)t}.$$

Solving

$$10000 = 200e^{(\ln(8)/3)t}$$

yields $t = (3\ln 50)/\ln 8 \approx 5.644$. The population is predicted to reach 10,000 bacteria in slightly more than five and a half hours.

Another example of proportional change is **Newton's Law of Cooling.** The laws of thermodynamics state that heat flows from areas of higher temperature to areas of lower temperature. A simple example is a hot object that cools down when placed in a cool room. Newton's Law of Cooling is the simple assumption that the temperature of the object changes at a rate proportional to the difference between the temperature of the object and the ambient temperature of the room. If T is the temperature of the object and A is the constant ambient temperature, Newton's Law of Cooling can be expressed as the differential equation

$$\frac{dT}{dt} = k(A - T).$$

This differential equation is both linear and separable. The separated form is

$$\frac{1}{A - T}\,dT = k\,dt.$$

Then an implicit definition of the temperature is given by

$$-\ln|A - T| = kt + C.$$

If we solve for T, we find the explicit temperature

$$T = A - Ce^{-kt}.$$

Notes:

Though we didn't show the steps, the explicit solution involves the typical process of renaming the constant $\pm e^{-C}$ as C, and allowing C to be positive, negative, or zero to account for both cases of the absolution value and to catch the constant solution $T = A$. Notice that the temperature of the object approaches the ambient temperature in the limit as $t \to \infty$.

Note: The equation $\frac{dT}{dt} = k(T - A)$ is also a valid representation of Newton's Law of Cooling. Intuition tells us that T will increase if T is less than A and decrease if T is greater then A. The form we use in the text follows this intuition with a positive k value. The form above will require that k take on a negative value. In the end, both forms result in the same function.

Example 6.4.2 Hot Coffee

A freshly brewed cup of coffee is set on the counter and has a temperature of 200° Fahrenheit. After 3 minutes, it has cooled to 190°, but is still too hot to drink. If the room is 72° and the coffee cools according to Newton's Law of Cooling, how long will the impatient coffee drinker have to wait until the coffee has cooled to 165°?

Solution Since we have already solved the differential equation for Newton's Law of Cooling, we can immediately use the function

$$T = A - Ce^{-kt}.$$

Since the room is 72°, we know $A = 72$. The initial temperature is 200°, which means $C = -128$. At this point, we have

$$T = 72 + 128e^{-kt}$$

The information about the coffee cooling to 190° in 3 minutes leads to the equation

$$190 = 72 + 128e^{-3k}.$$

Solving the exponential equation for k, we have

$$k = -\frac{1}{3}\ln\left(\frac{59}{64}\right) \approx 0.0271.$$

Finally, we finish the problem by solving the exponential equation

$$165 = 72 + 128e^{\frac{1}{3}\ln\left(\frac{59}{64}\right)t}.$$

The coffee drinker must wait $t = \dfrac{3\ln\left(\frac{93}{128}\right)}{\ln\left(\frac{59}{64}\right)} \approx 11.78$ minutes.

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We finish our discussion of models of proportional change by exploring three different models of disease spread through a population. In all of the models, we let y denote the proportion of the population that is sick ($0 \leq y \leq 1$). We assume a proportion of 0.05 is initially sick and that a proportion of 0.1 is sick 1 week later.

Example 6.4.3 Disease Spread 1

Suppose a disease spreads through a population at a rate proportional to the number of individuals who are sick. If 5% of the population is sick initially and 10% of the population is sick one week later, find a formula for the proportion of the population that is sick at time t.

Solution The assumption here seems to have some merit because it matches our intuition that a disease should spread more rapidly when more individuals are sick. The differential equation is simply

$$\frac{dy}{dt} = ky,$$

with solution

$$y = Ce^{kt}.$$

The conditions $y(0) = 0.05$ and $y(1) = 0.1$ lead to $C = 0.05$ a and $k = \ln 2$, so the function is

$$y = 0.05e^{(\ln 2)t}.$$

We should point out a glaring problem with this model. The variable y is a proportion and should take on values between 0 and 1, but the function $y = 0.05e^{2t}$ grows without bound. After $t \approx 4.32$ weeks, y exceeds 1, and the model ceases to make physical sense.

Example 6.4.4 Disease Spread 2

Suppose a disease spreads through a population at a rate proportional to the number of individuals who are not sick. If 5% of the population is sick initially and 10% of the population is sick one week later, find a formula for the proportion of the population that is sick at time t

Solution The intuition behind the assumption here is that a disease can only spread if there are individuals who are susceptible to the infection. As fewer and fewer people are able to be infected, the disease spread should slow down. Since y is proportion of the population that is

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sick, $1-y$ is the proportion who are not sick, and the differential equation is

$$\frac{dy}{dt} = k(1-y).$$

Though the context is quite different, the differential equation is identical to the differential equation for Newton's Law of Cooling, with $A = 1$. The solution is

$$y = 1 - Ce^{-kt}.$$

The conditions $y(0) = 0.05$ and $y(1) = 0.1$ yield $C = 0.95$ and $k = -\ln\left(\frac{18}{19}\right) \approx 0.0541$, so the final function is

$$y = 1 - .95e^{\ln\left(\frac{18}{19}\right)t}.$$

Notice that this function approaches $y = 1$ in the limit as $t \to \infty$, and does not suffer from the non-physical behavior described in Example 6.4.3.

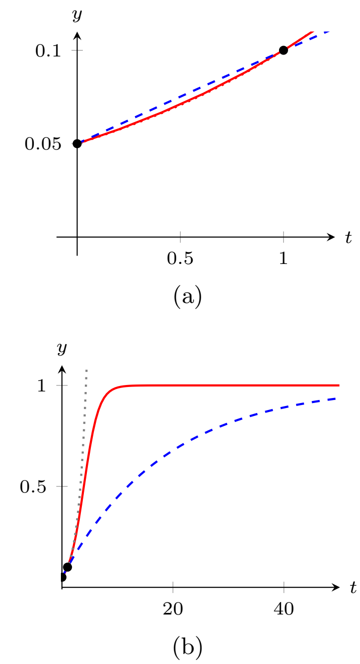

Figure 6.12: Plots of the functions from Example 6.4.3 (dotted), Example 6.4.4 (dashed), and Example 6.4.5 (solid).

In Example 6.4.3, we assumed disease spread depends on the number of infected individuals. In Example 6.4.4, we assumed disease spread depends on the number of susceptible individuals who are able to become infected. In reality, we would expect many diseases to require the interaction of both infected and susceptible individuals in order to spread. One of the simplest ways to model this required interaction is to assume disease spread depends on the product of the proportions of infected and uninfected individuals. This assumption (regularly seen in the context of chemical reactions) is often called the *law of mass action.*

Example 6.4.5 Disease Spread 3

Suppose a disease spreads through a population at a rate proportional to the product of the number of infected and uninfected individuals. If 5% of the population is sick initially and 10% of the population is sick one week later, find a formula for the proportion of the population that is sick at time t

Solution The differential equation is

$$\frac{dy}{dt} = ky(1-y).$$

This is exactly the logistic equation with $M = 1$. We solved this differential equation in Example 6.2.4, and found

$$y = \frac{1}{1 + be^{-kt}}.$$

Notes:

The conditions $y(0) = 0.05$ and $y(1) = 0.1$ yield $b = 19$ and $k = -\ln\left(\frac{9}{19}\right) \approx 0.7472$. The final function is

$$y = \frac{1}{1 + 19e^{\ln\left(\frac{9}{19}\right)t}}.$$

Based on the three different assumptions about the rate of disease spread explored in the last three examples, we now have three different functions giving the proportion of a population that is sick at time t. Each of the three functions meets the conditions $y(0) = 0.05$ and $y(1) = 0.1$. The three functions are shown in Figure 6.12. Notice that the logistic function mimics specific parts of the functions from Examples 6.4.3 and 6.4.4. We see in Figure 6.12(a) that the logistic and exponential functions are virtually indistinguishable for small t values. When there are few infected individuals and lots of susceptible individuals, the spread of a disease is largely determined by the number of sick people. The logistic curve captures this feature, and is "almost exponential" early on. In Figure 6.12(b), we see that the logistic curve leaves the exponential curve from Example 6.4.3 and approaches the curve from Example 6.4.4. This result implies that when most of the population is sick, the spread of the disease is largely dependent on the number of susceptible individuals. Though there are much more sophisticated mathematical models describing the spread of infections, we could argue that the logistic model presented in this example is the "best" of the three.

Rate-in Rate-out Problems

One of the classic ways to build a mathematical model involves tracking the way the amount of something can change. We sometimes say these models are based on *conservation laws*. Consider a box with some amount of a specific type of material inside. (Some type of chemical, for example.) The amount of material of the specific type in the box can only change in four ways; we can add more to the box, we can remove some from the box, some of the material can change into material of a different type, or some other type of material can turn into the type we're tracking. In the examples that follow, we assume material doesn't change type, so we only need to keep track of material coming into the box and material leaving the box. To derive a differential equation, we track rates:

$$\text{rate of change of some quantity} = \text{rate in} - \text{rate out}.$$

Though we stick to relatively simple examples, this basic idea can be used to derive some very important differential equations in mathematics and physics.

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The examples to follow involve tracking the amount of a chemical in solution. We assume liquid containing some chemical flows into a container at some rate. That liquid mixes instantaneously with the liquid already in the container. Then the liquid from the container flows out at some (potentially different) rate.

Note: The assumption about instantaneous mixing, though not physically accurate, leads to a differential equation we have hope of solving. In reality, the amount of chemical at a specific location in the container depends both on the location and how long we have been waiting. This dependence on both space and time leads to a type of differential equation called a *partial differential equation.* Differential equations of this type are more interesting, but significantly harder to study. Instantaneous mixing removes any spatial dependence from the problem, and leaves us with an *ordinary differential equation.*

Example 6.4.6 Equal Flow Rates

Suppose a 10 liter tank has 5 liters of salt solution in it. The initial concentration of the salt solution is 1 gram per liter. A salt solution with concentration 3 g/L flows into the tank at a rate of 2 L/min. Suppose the salt solution mixes instantaneously with the solution already in the tank and that the mixed solution from the tank flows out at a rate of 2 L/min. Find a function that gives the amount of salt in the tank at time t.

Solution We use the rate in − rate out setup described above. The quantity here is the amount (in grams) of salt in the tank at time t. Let y denote the amount of salt. In words, the differential equation is given by

$$\frac{dy}{dt} = \text{rate in} - \text{rate out}.$$

Thinking in terms of units can help fill in the details of the differential equation. Since y has units of grams, the left hand side of the equation has units g/min. Both terms on the right hand side must have these same units. Notice that the product of a concentration (with units g/L) and a flow rate (with units L/min) results in a quantity with units g/min. Both terms on the right hand side of the equation will include a concentration multiplied by a flow rate.

For the rate in, we multiply the inflow concentration by the rate that fluid is flowing into the bucket. This is $\left(3\frac{\text{g}}{\text{L}}\right)\left(2\frac{\text{L}}{\text{min}}\right) = 6$ g/min.

The rate out is more complicated. The flow rate is still 2 L/min, meaning that the overall volume of the fluid in the bucket is the constant

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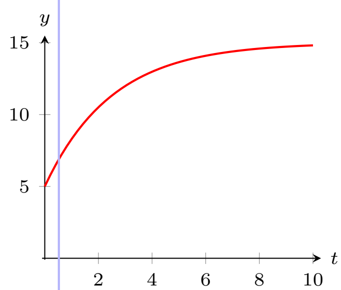

Figure 6.13: Salt concentration at time t, from Example 6.4.6.

5 L. The salt concentration in the bucket is not constant though, meaning that the outflow concentration is not constant. In particular, the outflow concentration is *not* the constant 1 g/L. This is simply the initial concentration. To find the concentration at any time, we need the amount of salt in the bucket at that time and the volume of liquid in the bucket at that time. The volume of liquid is the constant 5 L, and the amount of salt is given by the dependent variable y. Thus, the outflow concentration is $\frac{y}{5}$ g/L, yielding a rate out given by $\left(\frac{y}{5}\frac{\text{g}}{\text{L}}\right)\left(2\frac{\text{L}}{\text{min}}\right) = \frac{2y}{5}$ g/min

The differential equation we wish to solve is given by

$$\frac{dy}{dt} = 6 - \frac{2y}{5}.$$

To furnish an initial condition, we must convert the initial salt concentration into an initial amount of salt. This is $\left(1\frac{\text{g}}{\text{L}}\right)(5\text{ L}) = 5$ g, so $y(0) = 5$ is our initial condition.

Our differential equation is both separable and linear. We solve using separation of variables. The separated form of the differential equation is

$$\frac{5}{30 - 2y}\,dy = dt.$$

Integration yields the implicit solution

$$-\frac{5}{2}\ln|30 - 2y| = t + C.$$

Solving for y (and redefining the arbitrary constant C as necessary) yields the explicit solution

$$y = 15 + Ce^{-\frac{2}{5}t}.$$

The initial condition $y(0) = 5$ means that $C = -10$ so that

$$y = 15 - 10e^{-\frac{2}{5}t}$$

is the particular solution to our initial value problem.

This function is plotted in Figure 6.13. Notice that in the limit as $t \to \infty$, y approaches 15. This corresponds to a bucket concentration of $15/5 = 3$ g/L. It should not be surprising that salt concentration inside the tank will move to match the inflow salt concentration.

Notes:

Example 6.4.7 Unequal Flow Rates

Suppose the setup is identical to the setup in Example 6.4.6 except that now liquid flows out of the bucket at a rate of 1 L/min. Find a function that gives the amount of salt in the bucket at time t. What is the salt concentration when the solution ceases to be valid?

Solution Because the inflow and outflow rates no longer match, the volume of liquid in the bucket is not the constant 5 L. In general, we can find the volume of liquid via the equation

$$\text{volume} = \text{initial volume} + (\text{inflow rate - outflow rate})t.$$

In this example, the volume at time t is $5 + t$ liters. Because the total volume of the bucket is only 10 L, it follows that our solution will only be valid for $0 \leq t \leq 5$. At that point it is no longer possible to have liquid flow into a the bucket at a rate of 2 L/min and out of the bucket at a rate of 1 L/min.

To update the differential equation, we must modify the rate out. Since the volume is $5 + t$, the concentration at time t is given by $\frac{y}{5+t}$ g/L. Thus for rate out, we must use $\left(\frac{y}{5+t}\right)(1)$ g/min. The initial value problem is

$$\frac{dy}{dt} = 6 - \frac{y}{5+t}, \text{ with } y(0) = 5.$$

Unlike Example 6.4.6, where we had equal flow rates, this differential equation is no longer separable. We must proceed with an integrating factor. Writing the differential equation in the form

$$\frac{dy}{dt} + \frac{1}{5+t}y = 6,$$

we identify the integrating factor

$$\mu(t) = e^{\int \frac{1}{5+t}\,dt} = e^{\ln(5+t)} = 5 + t.$$

Then

$$\frac{d}{dt}\big((5+t)y\big) = 6(5+t),$$

yielding the implicit solution

$$(5+t)y = 30t + 3t^2 + C.$$

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The initial condition $y(0) = 5$ implies $C = 25$, so the explicit solution to our initial value problem is given by

$$y = \frac{3t^2 + 30t + 25}{5 + t}.$$

This solution ceases to be valid at $t = 5$. At that time, there are 25 g of salt in the tank. The volume of liquid is 10 L, resulting in a salt concentration of 2.5 g/L.

Differential equations are powerful tools that can be used to help describe the world around us. Though relatively simple in concept, the ideas of proportional change and matching rates can serve as building blocks in the development of more sophisticated mathematical models. As we saw in this section, some simple mathematical models can be solved analytically using the techniques developed in this chapter. Most sophisticated mathematical models don't allow for analytic solutions. Even so, there are an array of graphical and numerical techniques that can be used to analyze the model to make predictions and infer information about real world phenomena.

Notes:

Exercises 6.4

Problems

In Exercises 1 – 12, use the tools in the section to answer the questions presented.

1. Suppose the rate of change of y with respect to x is proportional to $10 - y$. Write down and solve a differential equation for y.

2. A rumor is spreading through a middle school with 250 students. Suppose the rumor spreads at a rate proportional to the number of students who haven't heard the rumor yet. If 1 person starts the rumor, and 75 students have heard the rumor 3 days later, how many days will it take until 80% of the students in the school have heard the rumor?

3. A rumor is spreading through a middle school with 250 students. Suppose the rumor spreads at a rate proportional to the product of number of students who have heard the rumor and the number who haven't heard the rumor. If 1 person starts the rumor, and 75 students have heard the rumor 3 days later, how many days will it take until 80% of the students in the school have heard the rumor?

4. A feature of radioactive decay is that the amount of a radioactive substance decreases at a rate proportional to the current amount of the substance. The *half life* of a substance is the amount of time it takes for half of a given amount of substance to decay. The half life of carbon-14 is approximately 5730 years. If an ancient object has a carbon-14 amount that is 20% of the original amount, how old is the object?

5. Consider a chemical reaction where molecules of type A combine with molecules of type B to form molecules of type C. Suppose one molecule of type A combines with one molecule of type B to form one molecule of type C, and that type C is produced at a rate proportional the product of the remaining number of molecules of types A and B. Let x denote moles of molecules of type C. Find a function giving the number of moles of type C at time t if there are originally a moles of type A, b moles of type B, and zero moles of type C.

6. Suppose an object with a temperature of 100° is introduced into a room with an ambient temperature of 70°. Suppose the temperature of the object changes at a rate proportional to the difference between the temperature of the object and the temperature of the room (Newton's Law of Cooling). If the object has cooled to 92° in 10 minutes, how long until the object has cooled to 84°?

7. Suppose an object with a temperature of 100° is introduced into a room with an ambient temperature given by $60 + 20e^{-\frac{1}{4}t}$ degrees. Suppose the temperature of the object changes at a rate proportional to the difference between the temperature of the object and the temperature of the room (Newton's Law of Cooling). If the object is 80° after 20 minutes, find a formula giving the temperature of the object at time t. (Note: This problem requires a numerical technique to solve for the unknown constants.)

8. A tank contains 5 gallons of salt solution with concentration 0.5 g/gal. Pure water flows into the tank at a rate of 1 gallon per minute. Salt solution flows out of the tank at a rate of 1 gallon per minute. (Assume instantaneous mixing.) Find the concentration of the salt solution at 10 minutes.

9. Dead leaves accumulate on the ground at a rate of 4 grams per square centimeter per year. The dead leaves on the ground decompose at a rate of 50% per year. Find a formula giving grams per square centimeter on the ground if there are no leaves on the ground at time $t = 0$.

10. A pond initially contains 10 million gallons of fresh water. Water containing an undesirable chemical flows into the pond at a rate of 5 million gallons per year, and fluid from the pond flows out at the same rate. (Assume instantaneous mixing.) If the concentration (in grams per million gallons) of the incoming chemical varies periodically according to the expression $2 + \sin(2t)$, find a formula giving the amount of chemical in the pond at time t.

11. A large tank contains 1 gallon of a salt solution with concentration 2 g/gal. A salt solution with concentration 1 g/gal flows into the tank at a rate of 4 gal/min. Salt solution flows out of the tank at a rate of 3 gal/min. (Assume instantaneous mixing.) Find the amount of salt in the tank at 10 minutes.

12. A stream flows into a pond containing 2 million gallons of fresh water at a rate of 1 million gallons per day. The stream flows out of the first pond and into a second pond containing 3 million gallons of fresh water. The stream then flows out of the second pond. Suppose the inflow and outflow rates are the same so that both ponds maintain their volumes. A factory upstream of the first pond starts polluting the stream. Directly below the factory, pollutant has a concentration of 55 grams per million gallons, and this concentration starts to flow into the first pond. Find the concentration of pollutant in the first and second ponds at 5 days.

: Solutions To Selected Problems

Chapter 1

Section 1.1

1. Answers will vary.
3. Answers will vary.
5. Answers will vary.
7. velocity
9. $1/9x^9 + C$
11. $t + C$
13. $-1/(3t) + C$
15. $2\sqrt{x} + C$
17. $-\cos\theta + C$
19. $5e^\theta + C$
21. $\frac{5^t}{2\ln 5} + C$
23. $t^6/6 + t^4/4 - 3t^2 + C$
25. $e^\pi x + C$
27. (a) $x > 0$
    (b) $1/x$
    (c) $x < 0$
    (d) $1/x$
    (e) $\ln|x| + C$. Explanations will vary.
29. $5e^x + 5$
31. $\tan x + 4$
33. $5/2x^2 + 7x + 3$
35. $5e^x - 2x$
37. $\frac{2x^4\ln^2(2)+2^x+x\ln 2)(\ln 32-1)+\ln^2(2)\cos(x)-1-\ln^2(2)}{\ln^2(2)}$
39. No answer provided.

Section 1.2

1. Answers will vary.
3. 0
5. (a) 3
   (b) 4
   (c) 3
   (d) 0
   (e) -4
   (f) 9
7. (a) 4
   (b) 2
   (c) 4
   (d) 2
   (e) 1
   (f) 2
9. (a) π
   (b) π
   (c) 2π
   (d) 10π
11. (a) $4/\pi$
    (b) $-4/\pi$
    (c) 0
    (d) $2/\pi$
13. (a) 40/3
    (b) 26/3
    (c) 8/3
    (d) 38/3
15. (a) 3ft/s
    (b) 9.5ft
    (c) 9.5ft
17. (a) 96ft/s
    (b) 6 seconds
    (c) 6 seconds
    (d) Never; the maximum height is 208ft.
19. 5
21. Answers can vary; one solution is $a = -2$, $b = 7$
23. -7
25. Answers can vary; one solution is $a = -11$, $b = 18$
27. $-\cos x - \sin x + \tan x + C$
29. $\ln|x| + \csc x + C$

Section 1.3

1. limits
3. Rectangles.
5. $2^2 + 3^2 + 4^2 = 29$
7. $0 - 1 + 0 + 1 + 0 = 0$
9. $-1 + 2 - 3 + 4 - 5 + 6 = 3$
11. $1 + 1 + 1 + 1 + 1 + 1 = 6$
13. Answers may vary; $\sum_{i=0}^{8}(i^2 - 1)$
15. Answers may vary; $\sum_{i=0}^{4}(-1)^i e^i$
17. 1045
19. -8525
21. 5050
23. 155
25. 24
27. 19
29. $\pi/3 + \pi/(2\sqrt{3}) \approx 1.954$
31. 0.388584
33. (a) Exact expressions will vary; $\frac{(1+n)^2}{4n^2}$.
    (b) 121/400, 10201/40000, 1002001/4000000
    (c) 1/4
35. (a) 8.
    (b) 8, 8, 8
    (c) 8
37. (a) Exact expressions will vary; $100 - 200/n$.

(b) 80, 98, 499/5

(c) 100

39. $F(x) = 5\tan x + 4$
41. $G(t) = 4/6t^6 - 5/4t^4 + 8t + 9$
43. $G(t) = \sin t - \cos t - 78$

Section 1.4

1. Answers will vary.
3. T
5. 20
7. 0
9. 1
11. $(5 - 1/5)/\ln 5$
13. -4
15. 16/3
17. 45/4
19. 1/2
21. 1/2
23. 1/4
25. 8
27. 0
29. Explanations will vary. A sketch will help.
31. $c = \pm 2/\sqrt{3}$
33. $c = 64/9 \approx 7.1$
35. $2/pi$
37. 16/3
39. $1/(e-1)$
41. 400ft
43. -1ft
45. -64ft/s
47. 2ft/s
49. 27/2
51. 9/2
53. $F'(x) = (3x^2+1)\frac{1}{x^3+x}$
55. $F'(x) = 2x(x^2+2) - (x+2)$

Chapter 2

Section 2.1

1. Chain Rule.
3. $\frac{1}{8}(x^3-5)^8 + C$
5. $\frac{1}{18}\left(x^2+1\right)^9 + C$
7. $\frac{1}{2}\ln|2x+7| + C$
9. $\frac{2}{3}(x+3)^{3/2} - 6(x+3)^{1/2} + C = \frac{2}{3}(x-6)\sqrt{x+3} + C$
11. $2e^{\sqrt{x}} + C$
13. $-\frac{1}{2x^2} - \frac{1}{x} + C$
15. $\frac{\sin^3(x)}{3} + C$
17. $-\tan(4-x) + C$
19. $\frac{\tan^3(x)}{3} + C$
21. $\tan(x) - x + C$
23. The key is to multiply $\csc x$ by 1 in the form $(\csc x + \cot x)/(\csc x + \cot x)$.
25. $\frac{e^{x^3}}{3} + C$
27. $x - e^{-x} + C$
29. $\frac{27^x}{\ln 27} + C$
31. $\frac{1}{2}\ln^2(x) + C$
33. $\frac{1}{6}\ln^2\left(x^3\right) + C$
35. $\frac{x^2}{2} + 3x + \ln|x| + C$
37. $\frac{x^3}{3} - \frac{x^2}{2} + x - 2\ln|x+1| + C$
39. $\frac{3}{2}x^2 - 8x + 15\ln|x+1| + C$
41. $\sqrt{7}\tan^{-1}\left(\frac{x}{\sqrt{7}}\right) + C$
43. $14\sin^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$
45. $\frac{5}{4}\sec^{-1}(|x|/4) + C$
47. $\frac{\tan^{-1}\left(\frac{x-1}{\sqrt{7}}\right)}{\sqrt{7}} + C$
49. $3\sin^{-1}\left(\frac{x-4}{5}\right) + C$
51. $-\frac{1}{3(x^3+3)} + C$
53. $-\sqrt{1-x^2} + C$
55. $-\frac{2}{3}\cos^{\frac{3}{2}}(x) + C$
57. $\frac{7}{3}\ln|3x+2| + C$
59. $\ln\left|x^2+7x+3\right| + C$
61. $-\frac{x^2}{2} + 2\ln\left|x^2-7x+1\right| + 7x + C$
63. $\tan^{-1}(2x) + C$
65. $\frac{1}{3}\sin^{-1}\left(\frac{3x}{4}\right) + C$
67. $\frac{19}{5}\tan^{-1}\left(\frac{x+6}{5}\right) - \ln\left|x^2+12x+61\right| + C$
69. $\frac{x^2}{2} - \frac{9}{2}\ln\left|x^2+9\right| + C$
71. $-\tan^{-1}(\cos(x)) + C$
73. $\ln|\sec x + \tan x| + C$ (integrand simplifies to $\sec x$)
75. $\sqrt{x^2-6x+8} + C$
77. 352/15
79. 1/5
81. $\pi/2$
83. $\pi/6$

Section 2.2

1. T
3. Determining which functions in the integrand to set equal to "u" and which to set equal to "dv".
5. $-e^{-x} - xe^{-x} + C$
7. $-x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$
9. $x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$
11. $1/2e^x(\sin x - \cos x) + C$
13. $1/13e^{2x}(2\sin(3x) - 3\cos(3x)) + C$

15. $-1/2\cos^2 x + C$

17. $x\tan^{-1}(2x) - \frac{1}{4}\ln\left|4x^2+1\right| + C$

19. $\sqrt{1-x^2} + x\sin^{-1} x + C$

21. $-\frac{x^2}{4} + \frac{1}{2}x^2\ln|x| + 2x - 2x\ln|x| + C$

23. $\frac{1}{2}x^2\ln\left(x^2\right) - \frac{x^2}{2} + C$

25. $2x + x\left(\ln|x|\right)^2 - 2x\ln|x| + C$

27. $x\tan(x) + \ln|\cos(x)| + C$

29. $\frac{2}{5}(x-2)^{5/2} + \frac{4}{3}(x-2)^{3/2} + C$

31. $\sec x + C$

33. $-x\csc x - \ln|\csc x + \cot x| + C$

35. $2\sin\left(\sqrt{x}\right) - 2\sqrt{x}\cos\left(\sqrt{x}\right) + C$

37. $2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$

39. π

41. 0

43. 1/2

45. $\frac{3}{4e^2} - \frac{5}{4e^4}$

47. $1/5\left(e^{\pi} + e^{-\pi}\right)$

Section 2.3

1. F

3. F

5. $\frac{1}{4}\sin^4(x) + C$

7. $\frac{1}{6}\cos^6 x - \frac{1}{4}\cos^4 x + C$

9. $-\frac{1}{9}\sin^9(x) + \frac{3\sin^7(x)}{7} - \frac{3\sin^5(x)}{5} + \frac{\sin^3(x)}{3} + C$

11. $\frac{1}{2}\left(-\frac{1}{8}\cos(8x) - \frac{1}{2}\cos(2x)\right) + C$

13. $\frac{1}{2}\left(\frac{1}{4}\sin(4x) - \frac{1}{10}\sin(10x)\right) + C$

15. $\frac{1}{2}\left(\sin(x) + \frac{1}{3}\sin(3x)\right) + C$

17. $\frac{\tan^5(x)}{5} + C$

19. $\frac{\tan^6(x)}{6} + \frac{\tan^4(x)}{4} + C$

21. $\frac{\sec^5(x)}{5} - \frac{\sec^3(x)}{3} + C$

23. $\frac{1}{3}\tan^3 x - \tan x + x + C$

25. $\frac{1}{2}\left(\sec x\tan x - \ln|\sec x + \tan x|\right) + C$

27. $\frac{2}{5}$

29. 32/315

31. 2/3

33. 16/15

Section 2.4

1. backwards

3. (a) $\tan^2\theta + 1 = \sec^2\theta$

   (b) $9\sec^2\theta$.

5. $\frac{1}{2}\left(x\sqrt{x^2+1} + \ln|\sqrt{x^2+1} + x|\right) + C$

7. $\frac{1}{2}\left(\sin^{-1} x + x\sqrt{1-x^2}\right) + C$

9. $\frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln|x + \sqrt{x^2-1}| + C$

11. $x\sqrt{x^2+1/4} + \frac{1}{4}\ln|2\sqrt{x^2+1/4} + 2x| + C =$
$\frac{1}{2}x\sqrt{4x^2+1} + \frac{1}{4}\ln|\sqrt{4x^2+1} + 2x| + C$

13. $4\left(\frac{1}{2}x\sqrt{x^2-1/16} - \frac{1}{32}\ln|4x + 4\sqrt{x^2-1/16}|\right) + C =$
$\frac{1}{2}x\sqrt{16x^2-1} - \frac{1}{8}\ln|4x + \sqrt{16x^2-1}| + C$

15. $3\sin^{-1}\left(\frac{x}{\sqrt{7}}\right) + C$ (Trig. Subst. is not needed)

17. $\sqrt{x^2-11} - \sqrt{11}\sec^{-1}(x/\sqrt{11}) + C$

19. $\sqrt{x^2-3} + C$ (Trig. Subst. is not needed)

21. $-\frac{1}{\sqrt{x^2+9}} + C$ (Trig. Subst. is not needed)

23. $\frac{1}{18}\frac{x+2}{x^2+4x+13} + \frac{1}{54}\tan^{-1}\left(\frac{x+2}{2}\right) + C$

25. $\frac{1}{7}\left(-\frac{\sqrt{5-x^2}}{x} - \sin^{-1}(x/\sqrt{5})\right) + C$

27. $\pi/2$

29. $2\sqrt{2} + 2\ln(1+\sqrt{2})$

31. $9\sin^{-1}(1/3) + \sqrt{8}$ Note: the new lower bound is $\theta = \sin^{-1}(-1/3)$ and the new upper bound is $\theta = \sin^{-1}(1/3)$. The final answer comes with recognizing that $\sin^{-1}(-1/3) = -\sin^{-1}(1/3)$ and that $\cos\left(\sin^{-1}(1/3)\right) = \cos\left(\sin^{-1}(-1/3)\right) = \sqrt{8}/3$.

Section 2.5

1. rational

3. $\frac{A}{x} + \frac{B}{x-3}$

5. $\frac{A}{x-\sqrt{7}} + \frac{B}{x+\sqrt{7}}$

7. $3\ln|x-2| + 4\ln|x+5| + C$

9. $\frac{1}{3}(\ln|x+2| - \ln|x-2|) + C$

11. $-\frac{4}{x+8} - 3\ln|x+8| + C$

13. $-\ln|2x-3| + 5\ln|x-1| + 2\ln|x+3| + C$

15. $x + \ln|x-1| - \ln|x+2| + C$

17. $2x + C$

19. $-\frac{3}{2}\ln\left|x^2+4x+10\right| + x + \frac{\tan^{-1}\left(\frac{x+2}{\sqrt{6}}\right)}{\sqrt{6}} + C$

21. $2\ln|x-3| + 2\ln|x^2+6x+10| - 4\tan^{-1}(x+3) + C$

23. $\frac{1}{2}\left(3\ln\left|x^2+2x+17\right| - 4\ln|x-7| + \tan^{-1}\left(\frac{x+1}{4}\right)\right) + C$

25. $\frac{1}{2}\ln\left|x^2+10x+27\right| + 5\ln|x+2| - 6\sqrt{2}\tan^{-1}\left(\frac{x+5}{\sqrt{2}}\right) + C$

27. $5\ln(9/4) - \frac{1}{3}\ln(17/2) \approx 3.3413$

29. 1/8

Section 2.6

1. The interval of integration is finite, and the integrand is continuous on that interval.

3. converges; could also state < 10.

5. $p > 1$

7. $e^5/2$

9. 1/3

11. $1/\ln 2$

13. diverges

15. 1

17. diverges

19. diverges

21. diverges

23. 1

25. 0

27. $-1/4$

29. -1

31. diverges

33. $1/2$

35. converges; Limit Comparison Test with $1/x^{3/2}$.

37. converges; Direct Comparison Test with xe^{-x}.

39. converges; Direct Comparison Test with xe^{-x}.

41. diverges; Direct Comparison Test with $x/(x^2+\cos x)$.

43. converges; Limit Comparison Test with $1/e^x$.

Section 2.7

1. F

3. They are superseded by the Trapezoidal Rule; it takes an equal amount of work and is generally more accurate.

5. (a) 250
   (b) 250
   (c) 250

7. (a) $2+\sqrt{2}+\sqrt{3}\approx 5.15$
   (b) $2/3(3+\sqrt{2}+2\sqrt{3})\approx 5.25$
   (c) $16/3\approx 5.33$

9. (a) 0.2207
   (b) 0.2005
   (c) $1/5$

11. (a) $9/2(1+\sqrt{3})\approx 12.294$
   (b) $3+6\sqrt{3}\approx 13.392$
   (c) $9\pi/2\approx 14.137$

13. Trapezoidal Rule: 3.0241
   Simpson's Rule: 2.9315

15. Trapezoidal Rule: 3.0695
   Simpson's Rule: 3.14295

17. Trapezoidal Rule: 2.52971
   Simpson's Rule: 2.5447

19. Trapezoidal Rule: 3.5472
   Simpson's Rule: 3.6133

21. (a) $n=150$ (using $\max\big(f''(x)\big)=1$)
   (b) $n=18$ (using $\max\big(f^{(4)}(x)\big)=7$)

23. (a) $n=5591$ (using $\max\big(f''(x)\big)=300$)
   (b) $n=46$ (using $\max\big(f^{(4)}(x)\big)=24$)

25. (a) Area is 25.0667 cm^2
   (b) Area is 250,667 yd^2

Chapter 3

Section 3.1

1. T

3. Answers will vary.

5. $16/3$

7. π

9. $2\sqrt{2}$

11. 4.5

13. $2-\pi/2$

15. $1/6$

17. On regions such as $[\pi/6, 5\pi/6]$, the area is $3\sqrt{3}/2$. On regions such as $[-\pi/2, \pi/6]$, the area is $3\sqrt{3}/4$.

19. $5/3$

21. $9/4$

23. 1

25. 4

27. 219,000 ft^2

Section 3.2

1. T

3. Recall that "dx" does not just "sit there;" it is multiplied by $A(x)$ and represents the thickness of a small slice of the solid. Therefore dx has units of in, giving $A(x)\,dx$ the units of in^3.

5. $175\pi/3$ units^3

7. $\pi/6$ units^3

9. $35\pi/3$ units^3

11. $2\pi/15$ units^3

13. (a) $512\pi/15$
   (b) $256\pi/5$
   (c) $832\pi/15$
   (d) $128\pi/3$

15. (a) $104\pi/15$
   (b) $64\pi/15$
   (c) $32\pi/5$

17. (a) 8π
   (b) 8π
   (c) $16\pi/3$
   (d) $8\pi/3$

19. The cross–sections of this cone are the same as the cone in Exercise 18. Thus they have the same volume of $250\pi/3$ units^3.

21. Orient the solid so that the x-axis is parallel to long side of the base. All cross–sections are trapezoids (at the far left, the trapezoid is a square; at the far right, the trapezoid has a top length of 0, making it a triangle). The area of the trapezoid at x is $A(x)=1/2(-1/2x+5+5)(5)=-5/4x+25$. The volume is 187.5 units^3.

Section 3.3

1. T

3. F

5. $9\pi/2$ units^3

7. $\pi^2-2\pi$ units^3

9. $48\pi\sqrt{3}/5$ units^3

11. $\pi^2/4$ units^3

13. (a) $4\pi/5$

(b) $8\pi/15$
(c) $\pi/2$
(d) $5\pi/6$

15. (a) $4\pi/3$
(b) $\pi/3$
(c) $4\pi/3$
(d) $2\pi/3$

17. (a) $2\pi(\sqrt{2}-1)$
(b) $2\pi(1-\sqrt{2}+\sinh^{-1}(1))$

Chapter 4

Section 4.1

1. $PV = \int_5^{\infty} 200 \cdot e^{-rt}\, dt = 200 \cdot \lim_{s\to\infty} \left(-\frac{1}{r}e^{-rt}\right)\Big|_5^s =$
$200 \cdot \lim_{s\to\infty} \frac{e^{-5r}-e^{-rs}}{r} = 200 \cdot \frac{e^{-5r}}{r}$.
$1000 \cdot \left[\frac{1-e^{-5r}}{r}\right] = PV \Rightarrow 1000 \cdot \left[\frac{1-e^{-5r}}{r}\right] =$
$200 \cdot \frac{e^{-5r}}{r} \Rightarrow 5-5e^{-5r} = e^{-5r} \Rightarrow 6e^{-5r} = 5 \Rightarrow e^{-5r} =$
$\frac{5}{6} \Rightarrow e^{5r} = \frac{6}{5} \Rightarrow r = \frac{1}{5}\ln\left(\frac{6}{5}\right) \approx 3.64\%$.

3. The future value is
$FV = \int_0^{40} f(t)\, dt = \int_0^{40} (2000+400t)e^{0.09(40-t)}dt =$
$e^{3.6}\left[\frac{2000e^{-0.09t}}{-0.09}\right]_0^{40} + e^{3.6}\left[\frac{te^{-0.09t}}{-0.09} - \frac{e^{-0.09t}}{-0.09^2}\right]_0^{40}$
$= 2,371,230$. Note that we use the fact that
$\int e^{rt} = \frac{e^{rt}}{r} + c$ and $\int te^{rt} = \frac{te^{rt}}{r} - \frac{e^{rt}}{r^2} + c$ when evaluate this integral.
The answer is $2,371,230, which in current dollars assuming a 4% rate of inflation is $478,743 - enough to live on for a good number of years. The amount of principal deposited is $\int_0^{40}(2000+400t)dt = 400,000$, which leaves roughly two million of interest - the lion's share. Note that even the first $2000 deposited becomes $73,200.

5. The present value of saving is
$\int_0^7 10,000e^{-0.10t}\, dt = \cdots = \$50,340$. The present value for maintenance and repair is
$\int_0^7 (1000+200t)e^{-0.10t}\, dt = \cdots = \$8149.60$. The present value of the scrap for $1000 in 7 years is
$1000e^{-0.10\cdot 7} = \$496.60$. So, the maximum amount that we should be willing to pay now is
$\$50,340 + \$8149.60 + \$496.60 = \$42,687$.

Section 4.2

1. a) $D(x) = S(x) \Rightarrow 100-0.05x = 10+0.1x \Rightarrow 0.15x = 90 \Rightarrow x = 600 \Rightarrow p = 100-0.05(600) = 70$. So, the equilibrium quantity and price is $(70, 600)$.

b) $CS = \int_0^{600}\Big[D(x)-70\Big]\, dx =$
$\int_0^{600}\Big[30-0.05x\Big]\, dx = 9000$ (dollars/unit), and
$PS = \int_0^{600}\Big[70-S(x)\Big]\, dx =$
$\int_0^{600}\Big[60-0.1x\Big]\, dx = 18,000$ (dollars/unit)

3. a) $D(x) = S(x) \Rightarrow e^{9-x} = e^{x+3} \Rightarrow 9-x = x+3 \Rightarrow 2x = 6 \Rightarrow x = 3 \Rightarrow p = e^6 \approx 403$. So, the equilibrium quantity and price is $(403, 3)$.

b) $CS = \int_0^3 \Big[D(x)-403\Big]\, dx =$
$\int_0^3 \Big[e^{9-x}-403\Big]\, dx = 45,432$ (dollars/unit), and
$PS = \int_0^3 \Big[403-S(x)\Big]\, dx = \int_0^3 \Big[403-e^{x+3}\Big]\, dx =$
... (dollars/unit).

5. a) $D(x) = S(x) \Rightarrow 20e^{-x} = 5e^{x} \Rightarrow e^{2x} = 4 \Rightarrow 2x = \ln 4 = 2\ln 2 \Rightarrow x = \ln 2 \Rightarrow p = 5e^{\ln 2} = 10$. So, the equilibrium quantity and price is $(10, \ln 2)$.

b) $CS = \int_0^{\ln 2}\Big[D(x)-10\Big]\, dx =$
$\int_0^{\ln 2}\Big[20e^{-x}-10\Big]\, dx = 3.07$ (dollars/unit), and
$PS = \int_0^{\ln 2}\Big[10-S(x)\Big]\, dx =$
$\int_0^{\ln 2}\Big[10-5e^{x}\Big]\, dx = 1.93$ (dollars/unit).

Section 4.3

1. $PV = \int_5^{\infty} 200 \cdot e^{-rt}\, dt = 200 \cdot \lim_{s\to\infty} \left(-\frac{1}{r}e^{-rt}\right)\Big|_5^s =$
$200 \cdot \lim_{s\to\infty} \frac{e^{-5r}-e^{-rs}}{r} = 200 \cdot \frac{e^{-5r}}{r}$.
$1000 \cdot \left[\frac{1-e^{-5r}}{r}\right] = PV \Rightarrow 1000 \cdot \left[\frac{1-e^{-5r}}{r}\right] =$
$200 \cdot \frac{e^{-5r}}{r} \Rightarrow 5-5e^{-5r} = e^{-5r} \Rightarrow 6e^{-5r} = 5 \Rightarrow e^{-5r} =$
$\frac{5}{6} \Rightarrow e^{5r} = \frac{6}{5} \Rightarrow r = \frac{1}{5}\ln\left(\frac{6}{5}\right) \approx 3.64\%$.

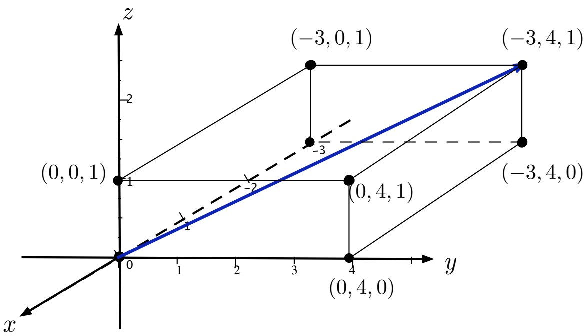
3. $FV = \int_0^{40} f(t)\, dt = \int_0^{40} (2000+400t)e^{0.09(40-t)}dt =$
$e^{3.6}\left[\frac{2000e^{-0.09t}}{-0.09}\right]_0^{40} + e^{3.6}\left[\frac{te^{-0.09t}}{-0.09} - \frac{e^{-0.09t}}{-0.09^2}\right]_0^{40}$
$= 2,371,230$. Note that we use the fact that
$\int e^{rt} = \frac{e^{rt}}{r} + c$ and $\int te^{rt} = \frac{te^{rt}}{r} - \frac{e^{rt}}{r^2} + c$ when evaluate this integral.
The answer is $2,371,230, which in current dollars assuming a 4% rate of inflation is $478,743 - enough to live on for a good number of years. The amount of principal deposited is $\int_0^{40}(2000+400t)dt = 400,000$, which leaves roughly two million of interest - the lion's share. Note that even the first $2000 deposited becomes $73,200.

5. The present value of saving is
$\int_0^7 10,000e^{-0.10t}\, dt = \cdots = \$50,340$. The present value for maintenance and repair is
$\int_0^7 (1000+200t)e^{-0.10t}\, dt = \cdots = \$8149.60$. The present value of the scrap for $1000 in 7 years is
$1000e^{-0.10\cdot 7} = \$496.60$. So, the maximum amount that we should be willing to pay now is
$\$50,340 + \$8149.60 + \$496.60 = \$42,687$.

Chapter 5

Section 5.1

1. Position now $(-3,4,1)$; distance travelled is $3+4+1=8$ units; distance from origin $= \sqrt{(-3)^2+4^2+1^2} = \sqrt{26} \approx 5.0990$ units.



3. a) $\left(4, -\frac{1}{2}, \frac{5}{2}\right)$.
   b) Take $P_1 \equiv (1,2,1)$; take $P_2 \equiv (7,-3,4)$.
   c) If $Q(2,7,-5)$ lies on the line through P_1 and P_2, then there is a common t-value that will give Q. Start with x-coordinate: $2 = 1+6t \Rightarrow t = 1/6$; next y-coordinate: $7 = 2-5t \Rightarrow t=-1$. Different t-values for each coordinate means Q is not collinear with P_1 and P_2.

5. Centre: $(-1/2,-1/2,-1/2)$; radius: $\sqrt{3}/2$.

7. Circle in yz-plane with : centre $(0,0)$; radius: 6.

9. Circle (point) in xy-plane with: centre $(0,0)$; radius: 0.

.

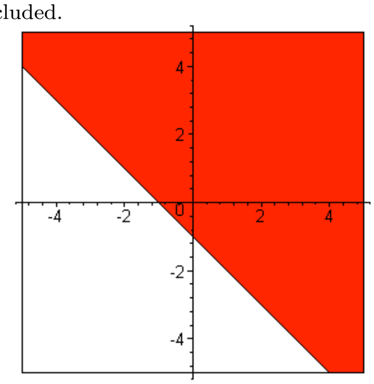
Section 5.2

1. a) TRUE: planes' orientation is determined by their normals (relative size and sign of each variable's coordinate); so if two normal are each parallel to a third normal, then they must be parallel to each other; so the planes are parallel. Without normals terminology: if two planes are parallel, then the coordinates for each variable are in a fixed ratio (planes $x+y+z=d_1$ and $-77x-77y-77z=d_2$ are parallel); therefore, if if planes #1 and #2 are each parallel to plane #3, then the ratio of coordinates for the three variables of #1 to those of #2 will also be in a fixed ratio.
   b) FALSE: consider planes $x+y+z=d_1$ and $y=d_2$: they are both perpendicular to the plane $x-z=d_3$ but are not parallel to each other. Notice, however, that the converse is TRUE: two parallel planes will be perpendicular to the same planes.
   c) FALSE: A plane parallel to a line means that the plane is parallel to a plane through the line (the plane can be moved - translated - so it contains the line); two non-parallel planes can certainly pass through the same line. For instance, planes $x+y+z=d_1$ and $y=d_2$ are both parallel to the line, passing through the origin, $x=z$, but as was argued in part (b), those planes are not parallel.
   d) TRUE: The quick answer is again using normal: the each plane's normal is perpendicular to that plane; so if two planes are perpendicular to one particular line, then those planes' normal must be parallel to the line, and so parallel to each other; and so the planes must be parallel.

3. The plane $5x-13y+z=-7$ passes through all three points - you can check by substituting each point.

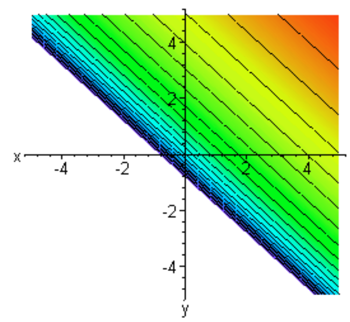
5. The associated points are $(0,3,3)$ and $(1,3,2)$.

7. a) The domain of f is all (x,y) such that $1+x+y>0$; that is, the line $y=(1+x)$ is the excluded boundary, and y-values should be larger than $(1+x)$; as shown in figure below, where the feasible region is shaded and its diagonal edge is not included.

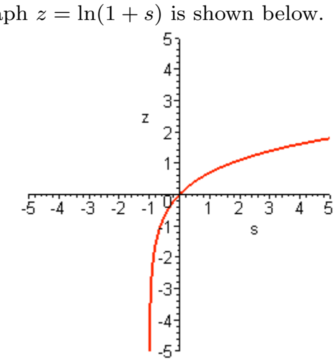


   b) The range of $\ln(1+x+y)$ is $\mathbb{R}$, all real numbers.

9. a) The contour map (lines of constant z-values) of $z=h(x,y)=\ln(1+x+y)$ is shown below. The lowest z-values have the deepest hue, toward the lower left edge; moving north east, as z-values increase, the graph gets lighter hued. The diagonal lines, parallel to the lowest line $1=x+y$, are the contour lines, each with a constant z-value. This observation is the key to understanding how the substitution $s=x+y$ gives the method of seeing that h is indeed a cylinder.



   b) The graph $z=\ln(1+s)$ is shown below.



   c) Notice, the graph in part (b) is identical to the graph of $z=\ln(1+x)$ in part 8(a), shown at right, only with the label on the horizontal axis (abscissa) changed from x to s. This observation gives the clue how to use the graph of $z = \ln(1+s)$ to generate the surface $z = \ln(1+x+y)$: as in 8(b), move the graph of $z = \ln(1+s)$ along the generating line, $0 = x + y$.

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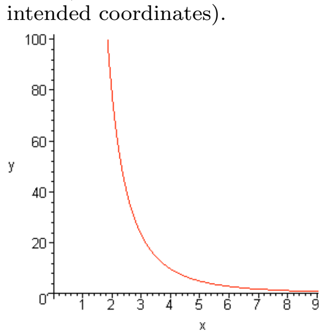
Section 5.3

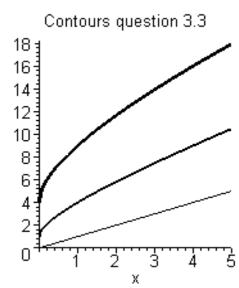
1. a) $z = 5.05 = 1.01x^{3/4}y^{1/4} \Rightarrow y^{1/4} = 5x^{-3/4} \Rightarrow y = 5^4 x^{-3} = 625x^{-3} = \dfrac{625}{x^3}$. This is the $z = 5.05$ iso-product curve (contour line).

   b) Setting $x = 5$ gives $y = \dfrac{625}{5^3} = 5$; noting the iso-product curve's formula can be written $5^4 = x^3 y$, another point could be $x = 25 (= 5^2)$ and $y = 5^{-2} = 0.04$ (which may have been the originally intended coordinates).

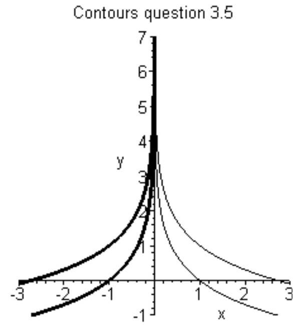


   c) The $z = 5.05$ iso-product curve is shown above.

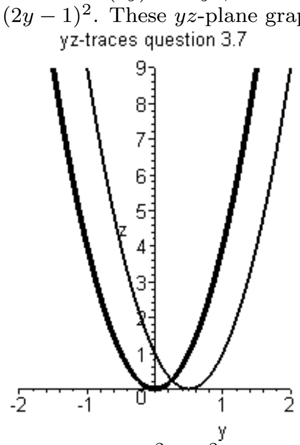
3. $z = y^{1/2} - x^{1/2} \Rightarrow y = (z+x)^2$, with $x, y > 0$. So for $z = 0, 1, 2$: $z = 0 \Rightarrow y = x, x > 0$; $z = 1 \Rightarrow y = \left(1 + x^{1/2}\right)^2$; and $z = 2 \Rightarrow y = \left(2 + x^{1/2}\right)^2$. The contours are in ascending order: the lowest (and thinnest) is for $z = 0$; the highest (and thickest) is for $z = 2$.



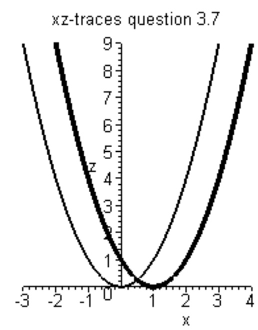
5. $z = xe^y \Rightarrow y = \ln(z/x) = \ln z - \ln x$, when x and z are positive, and $\ln(-z) - \ln(-x)$ when both are negative. For $z = 1 \Rightarrow y = -\ln x$; $z = -1 \Rightarrow y = -\ln(-x)$; $z = e \Rightarrow y = 1 - \ln x$; and $z = e \Rightarrow y = 1 - \ln(-x)$. The curves appear in symmetric pairs (reflected about the y-axis); the positive z-values, the first and third functions, are the thin curves on the right, positive side; the negative z-values, the second and fourth functions, are the thicker curves on the left, negative x-axis side; the lower curves are for the first two functions, with $|z| = 1$; and the upper curves are for the second two functions, with $|z| = e$.



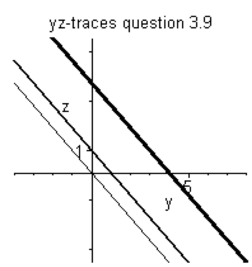
7. yz-traces: $x = 0 \Rightarrow z = (2y)^2 = 4y^2$; $x = 1 \Rightarrow z = (2y-1)^2$. These yz-plane graphs.



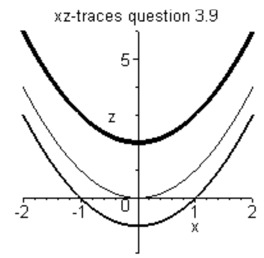
   xz-traces: $y = 0 \Rightarrow z = (-x)^2 = x^2$, $y = 1/2 \Rightarrow z = (1-x)^2$. These xz-plane graphs.



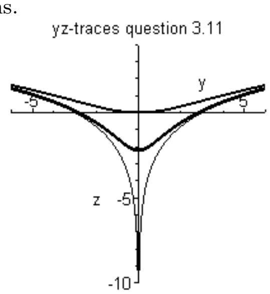
9. yz-traces: $x = 0 \Rightarrow z = -y$, the lower, thinnest line (passing through origin); $x = 1 \Rightarrow z = 1 - y$, middle line; $x = -2 \Rightarrow z = 4 - y$, upper, thickest line. These yz-plane graphs.



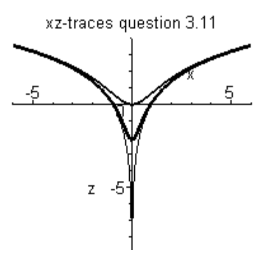
   xz-traces: $y = 0 \Rightarrow z = x^2$, the middle, thinnest line (tangent to x-axis at origin); $y = 1 \Rightarrow z = x^2 - 1$, the lowest line; $y = -2 \Rightarrow z = x^2 + 2$, the top, thickest line (with z intercept 2). These xz-plane graphs.



11. yz-traces: $x = 0 \Rightarrow z = 2(\ln y - \ln 3)$, the lowest, thinnest line with the z-axis its vertical asymptote ;
$x = 1 \Rightarrow z = \ln\left(1 + \dfrac{y^2}{9}\right)$, the top line tangent to the y-axis at the origin; $x = 1/3 \Rightarrow z = \ln(1 + y^2) - 2\ln 3$, the thickest line crossing the z-axis near $z = -2$. These yz-plane graphs.



xz-traces: $y = 0 \Rightarrow z = 2\ln x$, the lowest, thinnest line with the z-axis its vertical asymptote ;
$y = 1 \Rightarrow z = \ln(x^2 + 1/9)$, the thickest line crossing the z-axis near $z = 2$; $y = 3 \Rightarrow z = \ln(x^2 + 1)$ is the highest line tangent to the x-axis at the origin. These xz-plane graphs.



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Section 5.4

1. $f_x(x,y) = 100 \cdot \dfrac{2}{3} \cdot \left(\dfrac{y}{x}\right)^{1/3} \Rightarrow f_x(8,27) =$ $100 \cdot \dfrac{2}{3} \cdot \left(\dfrac{27}{8}\right)^{1/3} = 100 \cdot \dfrac{2}{3} \cdot \dfrac{3}{2} = 100$. That is, the MPL is \$100.00 additional production for each additional hour labour used.

3. a) $f_x(0,0) = 0$ [looks maybe < 0];

   b) $f_y(0,0) = 0$;

   c) $f_x(-2,0) < 0$;

   d) $f_y(-2,0) > 0$ [looks maybe $= 0$].

5. $f_x = -72x$; $f_y = -128y$; $f_{xx} = -72$; $fxy = 0$; $f_{yy} = -128$. Thus, we have $D = (-72)(-128) - (0)^2 = 210 \cdot 32 = 9216 (> 0)$.

7. $f_x = \dfrac{y}{x}$, $f_y = \ln x$, $f_{xx} = -\dfrac{y}{x^2}$, $f_{xy} = \dfrac{1}{x}$, $f_{yy} = 0$.
Thus, we have $D = \left(-\dfrac{y}{x^2}\right) \cdot \left(0\right) - \left(\dfrac{1}{x}\right)^2 = -\dfrac{1}{x^2} < 0$.

9. $f_x = ye^{x+y}$, $f_y = (1+y)e^{x+y}$, $f_{xx} = ye^{x+y}$, $f_{xy} = (1+y)e^{x+y}$, $f_{yy} = (2+y)e^{x+y}$. Thus, we have $D = \left(ye^{x+y}\right) \cdot \left((2+y)e^{x+y}\right) - \left((1+y)e^{x+y}\right)^2 = e^{2(x+y)}(-1) < 0$.

11. $f_x = \dfrac{y}{x^2+y^2}$, $f_y = \dfrac{-x}{x^2+y^2}$, $f_{xx} = \dfrac{-2xy}{(x^2+y^2)^2}$, $f_{xy} = \dfrac{x^2-y^2}{(x^2+y^2)^2}$, $f_{yy} = \dfrac{2xy}{(x^2+y^2)^2}$. Thus, we have $D = \left(\dfrac{-2xy}{(x^2+y^2)^2}\right) \cdot \left(\dfrac{2xy}{(x^2+y^2)^2}\right) - \left(\dfrac{x^2-y^2}{(x^2+y^2)^2}\right)^2 = -\dfrac{1}{(x^2+y^2)^2} < 0$.

13. $\dfrac{\partial z}{\partial x} = f'(xy) \cdot y \Rightarrow \left.\dfrac{\partial z}{\partial x}\right|_{1,1} = 5$, and
$\dfrac{\partial z}{\partial y} = f'(xy) \cdot x \Rightarrow \left.\dfrac{\partial z}{\partial y}\right|_{1,1} = 5$. Now,
$z_{xx} = f''(xy) \cdot y^2$, $z_{xy} = f'(xy) + f''(xy) \cdot xy$, $z_{yy} = f''(xy) \cdot x^2$. So,
$D(x,y) = -\left(f'(xy)\right)^2 - 2xyf'(xy)f''(xy) \Rightarrow D(1,1) = 5$.

15. a) $f(x,y) = ax^\alpha y^\beta \Rightarrow f_x = a\alpha x^{\alpha-1}y^\beta$, $f_y = a\beta x^\alpha y^{\beta-1}$, $f_{xx} = a\alpha(\alpha-1)x^{\alpha-2}y^\beta$,
$f_{xy} = a\alpha\beta x^{\alpha-1}y^{\beta-1}$, $f_{yy} = a\beta(\beta-1)x^\alpha y^{\beta-2}$.

    b) $D(x,y) = \left(a\alpha(\alpha-1)x^{\alpha-2}y^\beta\right) \cdot \left(a\beta(\beta-1)x^\alpha y^{\beta-2}\right) - \left(a\alpha\beta x^{\alpha-1}y^{\beta-1}\right)^2 = a^2\alpha\beta x^{2(\alpha-1)}y^{2(\beta-1)}\left(1-\alpha-\beta\right) = a^2x^{2(\alpha-1)}y^{2(\beta-1)}\{1-(\alpha+\beta)\}$. When, $\alpha+\beta = 1$, $D(x,y) = 0$ for all $(x,y) \in \mathbb{R}^2$.

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Section 5.5

1.
   - at $\left(0, 0, f(0,0) = 0\right)$, $T(x,y) = x + y$.
   - at $\left(1, 0, f(1,0) = \ln 2\right)$,
   $T(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(0,0)(y-0) = \ln 2 - \dfrac{1}{2} + \dfrac{1}{2}x + \dfrac{1}{2}y$.
   - at $\left(1, 1, f(1,1) = \ln 3\right)$,
   $T(x,y) = \ln 3 - \dfrac{2}{3} + \dfrac{1}{3}x + \dfrac{1}{3}y$.
   - at $\left(1, -1, f(1,-1) = 0\right)$, $T(x,y) = x + y$.

3. $T(x,y) = -2 + 4x$.
$f(0.9, 1.1) \approx T(0.9, 1.1) = -2 + 4(0.9) = 1.6$.

5. a) $D = (1)(1) - (-2)^2 = -3 < 0$. So, it is a saddle point.

   b) $D = (-1)(-2) - (1)^2 = 1 > 0$, $f_{xx} = -1 < 0$. So, it is a local maximum.

   c) $D = (-1)(1) - (0)^2 = -1 < 0$. So, it is a saddle point.

7.
   - $D(0,0) = 4 > 0$, $f_{xx}(0,0) = 2 > 0 \Rightarrow$ the point is a local minimum.

- $D\left(-\frac{2}{3},0\right) = -4 < 0 \Rightarrow$ the point is a saddle point.
- $D\left(0,\frac{2}{3}\right) = -4 < 0 \Rightarrow$ the point is a saddle point.
- $D\left(-\frac{2}{3},\frac{2}{3}\right) = 4 > 0,\ f_{xx}\left(-\frac{2}{3},\frac{2}{3}\right) < 0 \Rightarrow$ the point is a local maximum.

9. $D\left(-\frac{1}{2},4\right) = 3 > 0,\ f_{xx}\left(-\frac{1}{2},4\right) = -16 < 0 \Rightarrow$ the point is a local maximum.

11.
    - $D(0,0) = -36 < 0 \Rightarrow$ the point is a saddle point.
    - $D\left(2,2\right) = 108 > 0,\ f_{xx}\left(2,2\right) = 12 > 0 \Rightarrow$ the point is a local minimum.

13.
    - $D(0,0) = -36 < 0 \Rightarrow$ the point is a saddle point.
    - $D(3,1) = 180 > 0,\ f_{xx}(3,1) = -2 < 0 \Rightarrow$ the point is a local maximum.

15. The nearest point on the plane to P is $Q\left(-\frac{1}{14},\frac{4}{14},\frac{9}{14}\right)$. The is distance is $d = \sqrt{\left(-\frac{1}{14}-1\right)^2 + \left(\frac{4}{14}-1\right)^2 + \left(\frac{9}{14}-1\right)^2} \approx$ 1.33630621.

.

Section 5.6

1.
    - $f(0,1) = 1$, corresponding to $\lambda = \frac{1}{2}$.
    - $f(0,-1) = -1$, which is a minimum, corresponding to $\lambda = -\frac{1}{2}$.
    - $f\left(\pm\frac{\sqrt{3}}{2},\frac{1}{2}\right) = \frac{5}{4}$, which is a maximum, corresponding to $\lambda = 1$.

3. $f(4,7) = 30$, which is a minimum, corresponding to $\lambda = 2$.

5.
    a) $f\left(\frac{1}{4},\frac{1}{4}\right) = \frac{1}{2}$, corresponding to $\lambda = 1$.

    b) $f\left(\frac{1}{4},\frac{1}{4}\right) = \frac{1}{2}$, but $f(1,0) = f(0,1) = 1$. From $\sqrt{x}+\sqrt{y} = 1$, we can solve for y to be $y = (1-\sqrt{x})^2 \Rightarrow y' = 2(1-\sqrt{x})\cdot\left(-\frac{1}{2\sqrt{x}}\right) = 1 - x^{-1/2} \Rightarrow y'' = \frac{1}{2}x^{-3/2} > 0$.

    c) At this point, contour $z = \frac{1}{2}$ is a tangent to the constraint $\sqrt{x}+\sqrt{y} = 1$. Points $(0,1)$ and $(1,0)$ are corner point solutions (since $\sqrt{x}+\sqrt{y} = 1$ meets implicit constraint constraints $y \geq 0$ and $x \geq 0$.

.

Chapter 6

Section 6.1

1. An initial value problems is a differential equation that is paired with one or more initial conditions. A differential equation is simply the equation without the initial conditions.

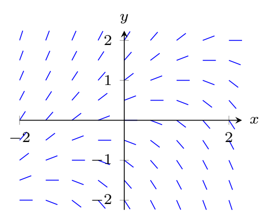
3. Substitute the proposed function into the differential equation, and show the the statement is satisfied.

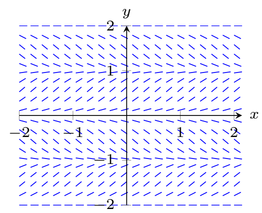
5. Many differential equations are impossible to solve analytically.

7. Answers will vary.

9. Answers will vary.

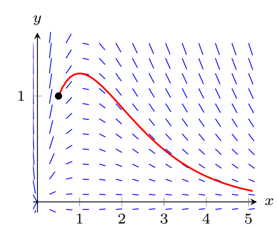
11. $C = 2$

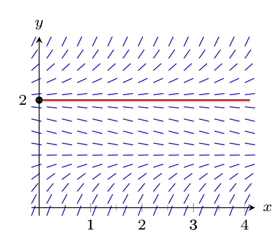
13. 

15. 

17. b

19. d

21. 

23. 

25.

x_i	y_i
0.00	1.0000
0.25	1.5000
0.50	2.3125
0.75	3.5938
1.00	5.5781

27.

x_i	y_i
0.0	2.0000
0.2	2.4000
0.4	2.9197
0.6	3.5816
0.8	4.4108
1.0	5.4364

29.

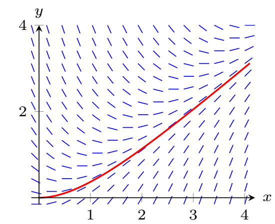
x_i	$y(x)$	h=0.2	$h=0.1$
0.0	1.0000	1.0000	1.0000
0.2	1.0204	1.0000	1.0100
0.4	1.0870	1.0400	1.0623
0.6	1.2195	1.1265	1.1687
0.8	1.4706	1.2788	1.3601
1.0	2.0000	1.5405	1.7129

Section 6.2

1. Separable. $\dfrac{1}{y^2-y}\,dy = dx$
3. Not separable.
5. $\left\{y = \dfrac{1+Ce^{2x}}{1-Ce^{2x}}, y=-1\right\}$
7. $y = Cx^4$
9. $(y-1)e^y = -e^{-x} - \dfrac{1}{3}e^{-3x} + C$
11. $\left\{\arcsin 2y - \arctan(x^2+1) = C, y = \pm\dfrac{1}{2}\right\}$
13. $\sin y + \cos x = 2$
15. $\frac{1}{2}y^2 - \ln(1+x^2) = 8$
17. $\dfrac{1}{2}y^2 - y = \dfrac{1}{2}\left((x^2+1)\ln(x^2+1) - (x^2+1)\right) + \dfrac{1}{2}$
19. $2\tan 2y = 2x + \sin 2x$

Section 6.3

1. $y = \dfrac{3}{2} + Ce^{2x}$
3. $y = -\dfrac{1}{2x} + Cx$
5. $y = \sec x + C(\csc x)$
7. $y = Ce^{3x} - (x+1)e^{2x}$
9. $y = (x^2+2)e^x$
11. $y = 1 - \dfrac{2}{x} + \dfrac{2-e^{1-x}}{x^2}$
13. $y = \dfrac{x^2+1}{x+1}e^{-x}$
15. $y = \dfrac{(x-2)(x+1)}{x-1}$
17. Both; $y = -5e^{x+\frac{1}{3}x^3}$
19. linear; $y = \dfrac{x^3-3x-6}{3(x-1)}$

21. 

The solution will increase and begin to follow the line $y = x-1$.
$y = x - 1 + e^{-x}$

Section 6.4

1. $y = 10 + Ce^{-kx}$
3. 4.43 days
5. $x = \begin{cases} \dfrac{ab(1-e^{(a-b)kt})}{b-ae^{(a-b)kt}} & \text{if } a \neq b \\ \dfrac{a^2kt}{1+akt} & \text{if } a = b \end{cases}$
7. $y = 60 - 3.69858e^{-\frac{1}{4}t} + 43.69858e^{-0.0390169t}$
9. $y = 8(1-e^{-\frac{1}{2}t})$ g/cm^2
11. 11.00075 g

Index

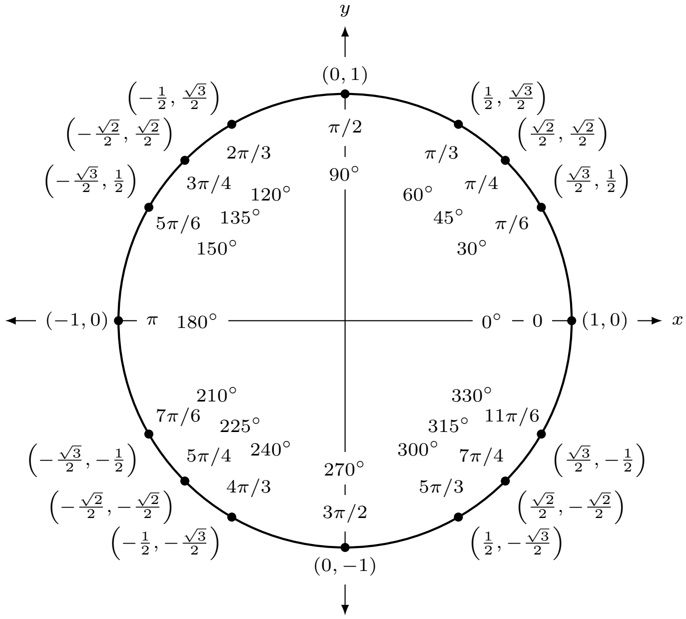
Differentiation Rules

1. $\frac{d}{dx}(cx) = c$
2. $\frac{d}{dx}(u \pm v) = u' \pm v'$
3. $\frac{d}{dx}(u \cdot v) = uv' + u'v$
4. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$
5. $\frac{d}{dx}(u(v)) = u'(v)v'$
6. $\frac{d}{dx}(c) = 0$
7. $\frac{d}{dx}(x) = 1$
8. $\frac{d}{dx}(x^n) = nx^{n-1}$
9. $\frac{d}{dx}(e^x) = e^x$
10. $\frac{d}{dx}(a^x) = \ln a \cdot a^x$
11. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
12. $\frac{d}{dx}(\log_a x) = \frac{1}{\ln a} \cdot \frac{1}{x}$
13. $\frac{d}{dx}(\sin x) = \cos x$
14. $\frac{d}{dx}(\cos x) = -\sin x$
15. $\frac{d}{dx}(\csc x) = -\csc x \cot x$
16. $\frac{d}{dx}(\sec x) = \sec x \tan x$
17. $\frac{d}{dx}(\tan x) = \sec^2 x$
18. $\frac{d}{dx}(\cot x) = -\csc^2 x$
19. $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$
20. $\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$
21. $\frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{|x|\sqrt{x^2-1}}$
22. $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}$
23. $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$
24. $\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$
25. $\frac{d}{dx}(\cosh x) = \sinh x$
26. $\frac{d}{dx}(\sinh x) = \cosh x$
27. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
28. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
29. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$
30. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
31. $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}}$
32. $\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2+1}}$
33. $\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = \frac{-1}{x\sqrt{1-x^2}}$
34. $\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = \frac{-1}{|x|\sqrt{1+x^2}}$
35. $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}$
36. $\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2}$

Integration Rules

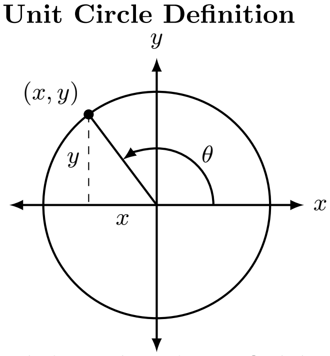
1. $\int c \cdot f(x)\, dx = c \int f(x)\, dx$
2. $\int f(x) \pm g(x)\, dx = \int f(x)\, dx \pm \int g(x)\, dx$
3. $\int 0\, dx = C$
4. $\int 1\, dx = x + C$
5. $\int x^n\, dx = \frac{1}{n+1} x^{n+1} + C,\ n \neq -1$
   $n \neq -1$
6. $\int e^x\, dx = e^x + C$
7. $\int a^x\, dx = \frac{1}{\ln a} \cdot a^x + C$
8. $\int \frac{1}{x}\, dx = \ln|x| + C$
9. $\int \cos x\, dx = \sin x + C$
10. $\int \sin x\, dx = -\cos x + C$
11. $\int \tan x\, dx = -\ln|\cos x| + C$
12. $\int \sec x\, dx = \ln|\sec x + \tan x| + C$
13. $\int \csc x\, dx = -\ln|\csc x + \cot x| + C$
14. $\int \cot x\, dx = \ln|\sin x| + C$
15. $\int \sec^2 x\, dx = \tan x + C$
16. $\int \csc^2 x\, dx = -\cot x + C$
17. $\int \sec x \tan x\, dx = \sec x + C$
18. $\int \csc x \cot x\, dx = -\csc x + C$
19. $\int \cos^2 x\, dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$
20. $\int \sin^2 x\, dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$
21. $\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
22. $\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
23. $\int \frac{1}{x\sqrt{x^2-a^2}}\, dx = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) + C$
24. $\int \cosh x\, dx = \sinh x + C$
25. $\int \sinh x\, dx = \cosh x + C$
26. $\int \tanh x\, dx = \ln(\cosh x) + C$
27. $\int \coth x\, dx = \ln|\sinh x| + C$
28. $\int \frac{1}{\sqrt{x^2-a^2}}\, dx = \ln\left|x + \sqrt{x^2-a^2}\right| + C$
29. $\int \frac{1}{\sqrt{x^2+a^2}}\, dx = \ln\left|x + \sqrt{x^2+a^2}\right| + C$
30. $\int \frac{1}{a^2-x^2}\, dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$
31. $\int \frac{1}{x\sqrt{a^2-x^2}}\, dx = \frac{1}{a}\ln\left(\frac{x}{a+\sqrt{a^2-x^2}}\right) + C$
32. $\int \frac{1}{x\sqrt{x^2+a^2}}\, dx = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{x^2+a^2}}\right| + C$

The Unit Circle



Definitions of the Trigonometric Functions

Unit Circle Definition

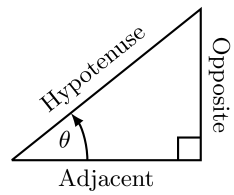


$\sin\theta = y \qquad \cos\theta = x$

$\csc\theta = \frac{1}{y} \qquad \sec\theta = \frac{1}{x}$

$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$

Right Triangle Definition



$\sin\theta = \frac{\text{O}}{\text{H}} \qquad \csc\theta = \frac{\text{H}}{\text{O}}$

$\cos\theta = \frac{\text{A}}{\text{H}} \qquad \sec\theta = \frac{\text{H}}{\text{A}}$

$\tan\theta = \frac{\text{O}}{\text{A}} \qquad \cot\theta = \frac{\text{A}}{\text{O}}$

Common Trigonometric Identities

Pythagorean Identities

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$1 + \cot^2 x = \csc^2 x$

Cofunction Identities

$\sin\left(\frac{\pi}{2} - x\right) = \cos x \qquad \csc\left(\frac{\pi}{2} - x\right) = \sec x$

$\cos\left(\frac{\pi}{2} - x\right) = \sin x \qquad \sec\left(\frac{\pi}{2} - x\right) = \csc x$

$\tan\left(\frac{\pi}{2} - x\right) = \cot x \qquad \cot\left(\frac{\pi}{2} - x\right) = \tan x$

Double Angle Formulas

$\sin 2x = 2\sin x\cos x$

$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x\\ &= 2\cos^2 x - 1\\ &= 1 - 2\sin^2 x\end{aligned}$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Sum to Product Formulas

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Even/Odd Identities

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\tan(-x) = -\tan x$

$\csc(-x) = -\csc x$

$\sec(-x) = \sec x$

$\cot(-x) = -\cot x$

Power–Reducing Formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

Product to Sum Formulas

$$\sin x \sin y = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$$

$$\cos x \cos y = \frac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$$

$$\sin x \cos y = \frac{1}{2}\big(\sin(x+y) + \sin(x-y)\big)$$

Angle Sum/Difference Formulas

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

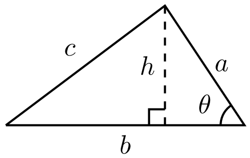
Areas and Volumes

Triangles

$h = a\sin\theta$

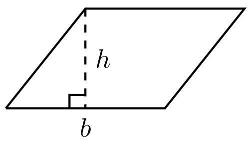
Area $= \frac{1}{2}bh$

Law of Cosines:
$c^2 = a^2 + b^2 - 2ab\cos\theta$



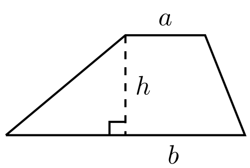
Parallelograms

Area $= bh$



Trapezoids

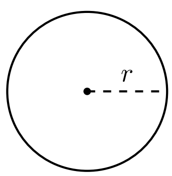
Area $= \frac{1}{2}(a+b)h$



Circles

Area $= \pi r^2$

Circumference $= 2\pi r$

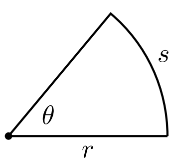


Sectors of Circles

θ in radians

Area $= \frac{1}{2}\theta r^2$

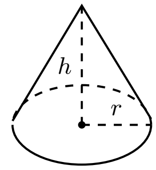
$s = r\theta$



Right Circular Cone

Volume $= \frac{1}{3}\pi r^2 h$

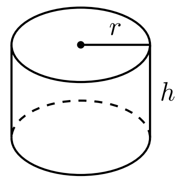
Surface Area $=$
$\pi r\sqrt{r^2+h^2} + \pi r^2$



Right Circular Cylinder

Volume $= \pi r^2 h$

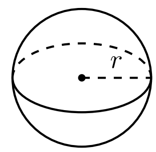
Surface Area $=$
$2\pi rh + 2\pi r^2$



Sphere

Volume $= \frac{4}{3}\pi r^3$

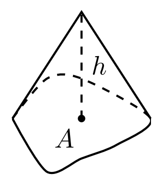
Surface Area $=4\pi r^2$



General Cone

Area of Base $= A$

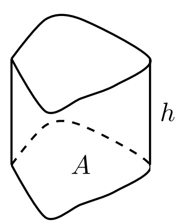
Volume $= \frac{1}{3}Ah$



General Right Cylinder

Area of Base $= A$

Volume $= Ah$



Algebra

Factors and Zeros of Polynomials

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial. If $p(a) = 0$, then a is a *zero* of the polynomial and a solution of the equation $p(x) = 0$. Furthermore, $(x - a)$ is a *factor* of the polynomial.

Fundamental Theorem of Algebra

An nth degree polynomial has n (not necessarily distinct) zeros. Although all of these zeros may be imaginary, a real polynomial of odd degree must have at least one real zero.

Quadratic Formula

If $p(x) = ax^2 + bx + c$, and $0 \le b^2 - 4ac$, then the real zeros of p are $x = (-b \pm \sqrt{b^2 - 4ac})/2a$

Special Factors

$x^2 - a^2 = (x - a)(x + a)$

$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$

$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$

$x^4 - a^4 = (x^2 - a^2)(x^2 + a^2)$

$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots + nxy^{n-1} + y^n$

$(x - y)^n = x^n - nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 - \cdots \pm nxy^{n-1} \mp y^n$

Binomial Theorem

$(x + y)^2 = x^2 + 2xy + y^2$

$(x - y)^2 = x^2 - 2xy + y^2$

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$(x - y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

Rational Zero Theorem

If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients, then every *rational zero* of p is of the form $x = r/s$, where r is a factor of a_0 and s is a factor of a_n.

Factoring by Grouping

$acx^3 + adx^2 + bcx + bd = ax^2(cs + d) + b(cx + d) = (ax^2 + b)(cx + d)$

Arithmetic Operations

$ab + ac = a(b + c)$

$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

$\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$

$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc}$

$\frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc}$

$\frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b}$

$a\left(\frac{b}{c}\right) = \frac{ab}{c}$

$\frac{a - b}{c - d} = \frac{b - a}{d - c}$

$\frac{ab + ac}{a} = b + c$

Exponents and Radicals

$a^0 = 1, \quad a \neq 0$

$(ab)^x = a^x b^x$

$a^x a^y = a^{x+y}$

$\sqrt{a} = a^{1/2}$

$\frac{a^x}{a^y} = a^{x-y}$

$\sqrt[n]{a} = a^{1/n}$

$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

$\sqrt[n]{a^m} = a^{m/n}$

$a^{-x} = \frac{1}{a^x}$

$\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$

$(a^x)^y = a^{xy}$

$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Additional Formulas

Summation Formulas:

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Trapezoidal Rule:

$$\int_a^b f(x)\ dx \approx \frac{\Delta x}{2}\left[f(x_1) + 2f(x_2) + 2f(x_3) + ... + 2f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^3}{12n^2}\left[\max\left|f''(x)\right|\right]$

Simpson's Rule:

$$\int_a^b f(x)\ dx \approx \frac{\Delta x}{3}\left[f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + ... + 2f(x_{n-1}) + 4f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^5}{180n^4}\left[\max\left|f^{(4)}(x)\right|\right]$

Arc Length:

$$L = \int_a^b \sqrt{1 + f'(x)^2}\ dx$$

Surface of Revolution:

$$S = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\ dx$$

(where $f(x) \geq 0$)

$$S = 2\pi \int_a^b x\sqrt{1 + f'(x)^2}\ dx$$

(where $a, b \geq 0$)

Work Done by a Variable Force:

$$W = \int_a^b F(x)\ dx$$

Force Exerted by a Fluid:

$$F = \int_a^b w\, d(y)\, \ell(y)\ dy$$

Taylor Series Expansion for $f(x)$:

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + ... + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Maclaurin Series Expansion for $f(x)$, where $c = 0$:

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ... + \frac{f^{(n)}(0)}{n!}x^n$$

Summary of Tests for Series:

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} r^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	Sum $= \frac{1}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+a})$	$\lim_{n\to\infty} b_n = L$		Sum $= \left(\sum_{n=1}^{a} b_n\right) - L$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$	$p > 1$	$p \leq 1$	
Integral Test	$\sum_{n=0}^{\infty} a_n$	$\int_1^{\infty} a(n)\ dn$ is convergent	$\int_1^{\infty} a(n)\ dn$ is divergent	$a_n = a(n)$ must be continuous
Direct Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$	$\sum_{n=0}^{\infty} b_n$ diverges and $0 \leq b_n \leq a_n$	
Limit Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $\lim_{n\to\infty} a_n/b_n \geq 0$	$\sum_{n=0}^{\infty} b_n$ diverges and $\lim_{n\to\infty} a_n/b_n > 0$	Also diverges if $\lim_{n\to\infty} a_n/b_n = \infty$
Ratio Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n\to\infty} a_{n+1}/a_n = \infty$
Root Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} \left(a_n\right)^{1/n} < 1$	$\lim_{n\to\infty} \left(a_n\right)^{1/n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n\to\infty} \left(a_n\right)^{1/n} = \infty$

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